Extended Supersymmetric σ-Model Based on the SO(2N+1) Lie Algebra of the Fermion Operators[*]

Seiya NISHIYAMA[†], João da PROVIDÊNCIA[‡]

Constança PROVIDÊNCIA[§] and Flávio CORDEIRO[¶]

Centro de Física Teórica, Departamento de Física,
Universidade de Coimbra
P-3004-516 Coimbra, Portugal

February 2, 2008

Abstract

Extended supersymmetric σ-model is given, standing on the $SO(2N+1)$ Lie algebra of fermion operators composed of annihilation-creation operators and pair operators. Canonical transformation, the extension of the $SO(2N)$ Bogoliubov transformation to the $SO(2N+1)$ group, is introduced. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables, we investigate a new aspect of the supersymmetric σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. We construct a Killing potential which is just the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space given by van Holten et al. to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. To our great surprise, the Killing potential is equivalent with the generalized density matrix. Its diagonal-block matrix is related to a reduced scalar potential with a Fayet-Ilipoulos term. The reduced scalar potential is optimized in order to see the behaviour of the vacuum expectation value of the σ-model fields and a proper solution for one of the $SO(2N+1)$ group parameters is obtained. We give bosonization of the $SO(2N+2)$ Lie operators, vacuum functions and differential forms for their bosons expressed in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, a $U(1)$ phase and the corresponding Kähler potential.

[*]A preliminary version of this work has been presented by S. Nishiyama at the YITP workshop YITP-W-07-05 on *String Theory and Quantum Field Theory* held at Kinki University, 6 - 10 August, 2007.

[†]Corresponding author. E-mail address: seikoaquarius@ybb.ne.jp, nisiyama@teor.fis.uc.pt

[‡]E-mail address: providencia@teor.fis.uc.pt

[§]E-mail address: cp@teor.fis.uc.pt

[¶]E-mail address: flaviocordeiro_704@hotmail.com

1 Introduction

The supersymmetric extension of nonlinear models was first given by Zumino under the introduction of scalar fields taking values in a complex Kähler manifold [1]. The higher dimensional nonlinear σ-models defined on symmetric spaces and on hyper Kähler manifolds have been itensively studied in various contexts in modern versions of elementary particle physics, superstring theory and supergravity theory [2, 3, 4, 5].

In nuclear and condensed matter physics, the time dependent Hartree-Bogoliubov (TDHB) theory [6, 7] has been regarded as the standard approximation in the many-body theoretical descriptions of superconducting fermion systems [8, 9]. In the TDHB an HB wavefunction (WF) for such systems represents Bose condensate states of fermion pairs. It is a good approximation for the ground state of the system with a short-range pairing interaction that produces a spontaneous Bose condensation of the fermion pairs. Number-nonconservation of the HB WF is a consequence of the spontaneous Bose condensation of fermion pairs. Standing on the Lie-algebraic viewpoint, the pair operators of fermion with N indices form the $SO(2N)$ Lie algebra and accompany the $U(N)$ Lie algebra as a sub-algebra. $SO(2N)(=g)$ and $U(N)(=h)$ denote the special orthogonal group of $2N$ dimensions and the unitary group of N dimensions, respectively. One can give an integral representation of a state vector on the group g, the exact coherent state representation (CS rep) of a fermion system [10]. It makes possible global approach to the above problem. The canonical transformation of the fermion operators generated by the Lie operators in the $SO(2N)$ Lie algebra induces the well-known generalized Bogoliubov transformation for the fermions. The TDHB equation has been derived from the Euler-Lagrange equation of motion (EOM) for the $\frac{g}{h}=\frac{SO(2N)}{U(N)}$ coset variables by one of the present authors (SN) [11]. The TDHB theory is, however, applicable only to even fermion systems. For odd fermion systems we have no TD self-consistent field (SCF) theory with the same level of the mean field (MF) approximation as the TDHB.

van Holten et al. have discussed a procedure for consistent coupling of gauge- and matter superfields to supersymmetric σ-models on the Kähler coset spaces. They have presented a way of constructing the Killing potentials and have applied their method to the explcit construction of supersymmetric σ-models on the coset spaces $\frac{SO(2N)}{U(N)}$. They have shown that only a finite number of the coset models can be consistent when coupled to matter superfields with $U(N)$ quantum numbers reflecting spinorial representations of $SO(2N)$ [2]. Higashijima et al. have given Ricci-flat metrics on compact Kähler manifolds, $\frac{SU(N)}{[SU(N-M)\times U(M)]}$, $\frac{SO(2N)}{U(N)}$ and $\frac{Sp(N)}{U(N)}$ and non-compact Kähler manifolds, applying their technique of the gauge theory formulation of supersymmetric nonlinear σ-models on the hermitian symmetric spaces [3]. Preceding these works, Deldug and Valent have investigated the Kählerian σ-models in two-dimensional space-time at the classical quantum level. They have presented a unified treatment of the models based on irreducible hermitian symmetric spaces corresponding to the coset spaces $\frac{G}{H}$ [12]. van Holten has also discussed the construction of σ-models on compact and non-compact Grassmannian manifolds, $\frac{SU(N+M)}{S[U(N)\times U(M)]}$ and $\frac{SU(N,M)}{S[U(N)\times U(M)]}$ [13].

One of the most challenging problems in the current studies of nuclear physics is to give a theory suitable for description of collective motions with large amplitudes in both even and odd soft nuclei with strong collective correlations. For providing the general microscopic means for a unified self-consistent description for Bose and Fermi type collective excitations in such fermion systems, Fukutome, Yamamura and one of the present authors (SN) have

proposed a new fermion many-body theory basing on the $SO(2N+1)$ Lie algebra of fermion operators [14]. The set of the fermion operators composed of creation-annihilation and pair operators forms a larger Lie algebra, the Lie algebra of the $SO(2N+1)$ group. A representation of an $SO(2N+1)$ group has been derived by a group extension of the $SO(2N)$ Bogoliubov transformation for fermions to a new canonical transformation group. The fermion Lie operators, when operated onto the integral representation of the $SO(2N+1)$ WF, are mapped into the regular representation of the $SO(2N+1)$ group and are represented by boson operators. The boson images of the fermion Lie operators are expressed by closed first order differential forms. The creation-annihilation operators themselves as well as the pair operators are given by the Schwinger-type boson representation as a natural consequence [15, 16].

Along the same way as the above, we give an extended supersymmetric σ-model on Kähler coset space $\frac{G}{H}=\frac{SO(2N+2)}{U(N+1)}$, basing on the $SO(2N+1)$ Lie algebra of the fermion operators. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables [17], we investigate a new aspect of the supersymmetric σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. We construct a Killing potential which is just the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space given by van Holten et al. [2] to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. To our great surprise, the Killing potential is equivalent with the generalized density matrix. Its diagonal-block part is related to a reduced scalar potential with a Fayet-Ilipoulos term. The reduced scalar potential is optimized in order to see the behaviour of the vacuum expectation value of the σ-model fields. We get, however, a too simple solution. The optimiztion of the reduced scalar potential plays an important role to evaluate the criteria for supersymmetry-breaking and internal symmetry-breaking. To find a proper solution for the extended supersymmetryic σ-mdoel, after rescaling Goldstone fields by a mass parameter, minimization of the deformed reduced scalar potential is also made.

Previously, using the above embedding we have developed an extended TDHB (ETDHB) theory, in which paired and unpaired modes are treated on the same footing [14, 11]. The ETDHB applicable to both even and odd fermion systems is a TDSCF with the same level of MF approximation as the usual TDHB for even fermion systems [9]. The ETDHB equation is derived from a classical Euler-Lagrange EOM for the $\frac{SO(2N+2)}{U(N+1)}$ coset variables.

In sect. 2, we recapitulate briefly an induced representation of an $SO(2N+1)$ canonical transformation group. In sect. 3, an embedding of the $SO(2N+1)$ group into an $SO(2N+2)$ group is made and introduction of $\frac{SO(2N+2)}{U(N+1)}$ coset variables is also made. In sect. 4, we give an extended supersymmetric σ-model on the coset space $\frac{SO(2N+2)}{U(N+1)}$ based on the $SO(2N+1)$ Lie algebra of the fermion operators and study a new aspect of the extended supersymmetric σ-model on the Kähler manifold, the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. In sect. 5, the expressions for Killing potentials in that coset space are given and their equivalence with the generalized density matrix is proved and then a reduced scalar potential is derived. Finally, in sect. 6, we give discussions on the optimized scalar potential and concluding remarks. In Appendix, we give a bosonization of the $SO(2N+2)$ Lie operators, vacuum functions and differential forms for their bosons expressed in terms of the corresponding Kähler potential, the $\frac{SO(2N+2)}{U(N+1)}$ coset variables and a $U(1)$ phase and make a brief sketch of derivation of the ETDHB equation from the Euler-Lagrange EOM for those coset variables in the TDSCF. Throughout this paper, we use the summation convention over repeated indices unless there is the danger of misunderstanding.

2 The SO(2N+1) Lie algebra of fermion operators and the Bogoliubov transformation

Let c_α and $c_\alpha^\dagger$, $\alpha=1,\cdots, N$, be annihilation and creation operators of the fermion satisfying the canonical anti-commutation relations

$$\{c_\alpha,\ c_\beta^\dagger\} = \delta_{\alpha\beta}, \quad \{c_\alpha,\ c_\beta\} = \{c_\alpha^\dagger,\ c_\beta^\dagger\} = 0. \tag{2.1}$$

We introduce the set of fermion operators consisting of the following annihilation and creation operators and pair operators:

$$\left.\begin{array}{l} c_\alpha,\ c_\alpha^\dagger, \\[2ex] E^\alpha_{\ \beta} = c_\alpha^\dagger c_\beta - \dfrac{1}{2}\delta_{\alpha\beta}, \quad E^{\alpha\beta} = c_\alpha^\dagger c_\beta^\dagger, \quad E_{\alpha\beta} = c_\alpha c_\beta, \\[2ex] E^{\alpha\dagger}_{\ \beta} = E^\beta_{\ \alpha}, \quad E^{\alpha\beta} = E^\dagger_{\beta\alpha}, \quad E_{\alpha\beta} = -E_{\beta\alpha}. \quad (\alpha,\beta = 1,\cdots,N) \end{array}\right\} \tag{2.2}$$

It is well known that the set of fermion operators (2.2) form an $SO(2N+1)$ Lie algebra. As a consequence of the anti-commutation relation (2.1), the commutation relations for the fermion operators (2.2) in the $SO(2N+1)$ Lie algebra are

$$[E^\alpha_{\ \beta},\ E^\gamma_{\ \delta}] = \delta_{\gamma\beta}E^\alpha_{\ \delta} - \delta_{\alpha\delta}E^\gamma_{\ \beta}, \quad (U(N) \text{ algebra}) \tag{2.3}$$

$$\left.\begin{array}{l} [E^\alpha_{\ \beta},\ E_{\gamma\delta}] = \delta_{\alpha\delta}E_{\beta\gamma} - \delta_{\alpha\gamma}E_{\beta\delta}, \\[2ex] [E^{\alpha\beta},\ E_{\gamma\delta}] = \delta_{\alpha\delta}E^\beta_{\ \gamma} + \delta_{\beta\gamma}E^\alpha_{\ \delta} - \delta_{\alpha\gamma}E^\beta_{\ \delta} - \delta_{\beta\delta}E^\alpha_{\ \gamma}, \\[2ex] [E_{\alpha\beta},\ E_{\gamma\delta}] = 0, \end{array}\right\} \tag{2.4}$$

$$\left.\begin{array}{l} [c_\alpha^\dagger,\ c_\beta] = 2E^\alpha_{\ \beta},\ [c_\alpha,\ c_\beta] = 2E_{\alpha\beta}, \\[2ex] [c_\alpha,\ E^\beta_{\ \gamma}] = \delta_{\alpha\beta}c_\gamma,\ [c_\alpha,\ E_{\beta\gamma}] = 0, \\[2ex] [c_\alpha,\ E^{\beta\gamma}] = \delta_{\alpha\beta}c_\gamma^\dagger - \delta_{\alpha\gamma}c_\beta^\dagger. \end{array}\right\} \tag{2.5}$$

We omit the commutation relations obtained by hermitian conjugation of (2.4) and (2.5). The $SO(2N+1)$ Lie algebra of the fermion operators contains the $U(N)(=\{E^\alpha_{\ \beta}\})$ and the $SO(2N)(=\{E^\alpha_{\ \beta},\ E^{\alpha\beta},\ E_{\alpha\beta}\})$ Lie algebras of the pair operators as sub-algebras.

An $SO(2N)$ canonical transformation $U(g)$ is generated by the fermion $SO(2N)$ Lie operators. The $U(g)$ is the generalized Bogoliubov transformation [7] specified by an $SO(2N)$ matrix g

$$U(g)(c, c^\dagger)U^\dagger(g) = (c, c^\dagger)g, \tag{2.6}$$

$$g \stackrel{\text{def}}{=} \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix}, \qquad g^\dagger g = gg^\dagger = 1_{2N}, \qquad \det g = 1, \tag{2.7}$$

$$U(g)U(g') = U(gg'), \quad U(g^{-1}) = U^{-1}(g) = U^\dagger(g), \quad U(1_{2N}) = \mathbb{I}_g \text{ (unit operator on } g), \tag{2.8}$$

where $(c,\, c^\dagger)$ is the $2N$-dimensional row vector $((c_\alpha),\, (c^\dagger_\alpha))$ and $a=(a^\alpha{}_\beta)$ and $b=(b_{\alpha\beta})$ are $N\times N$ matrices. The bar denotes the complex conjugation. The HB ($SO(2N)$) WF $|\,g\!>$ is generated as $|\,g\!>=U(g)|\,0\!>$ where $|\,0\!>$ is the vacuum satisfying $c_\alpha|\,0\!>=0$. The matrix g is composed of the matrices a and b satisfying the ortho-normalization condition. The $|\,g\!>$ is expressed as

$$|\,g\!> = <\!0\,|U(g)|\,0\!>\exp(\frac{1}{2}\cdot q_{\alpha\beta}c^\dagger_\alpha c^\dagger_\beta)|\,0\!>, \tag{2.9}$$

$$<\!0\,|U(g)|\,0\!> = \bar{\Phi}_{00}(g) = [\det(a)]^{\frac{1}{2}} = \left[\det(1_N + q^\dagger q)\right]^{-\frac{1}{4}} e^{i\frac{\tau}{2}}\ , \tag{2.10}$$

$$q = ba^{-1} = -q^{\mathrm{T}},\ (\text{variables of the } \tfrac{SO(2N)}{U(N)} \text{ coset space}),\ \tau = \frac{i}{2}\ln\left[\frac{\det(a^*)}{\det(a)}\right], \tag{2.11}$$

where det means determinant and the symbol T denotes the transposition.

The canonical anti-commutation relation gives us not only the above Lie algebras but also the other three algebras. Let n be the fermion number operator $n=c^\dagger_\alpha c_\alpha$. The operator $(-1)^n$ anticommutes with c_α and $c^\dagger_\alpha$;

$$\{c_\alpha,\ (-1)^n\} = \{c^\dagger_\alpha,\ (-1)^n\} = 0. \tag{2.12}$$

Let us introduce the operator Θ defined by $\Theta\equiv\theta_\alpha c^\dagger_\alpha - \bar{\theta}_\alpha c_\alpha$. Due to the relation $\Theta^2=-\bar{\theta}_\alpha\theta_\alpha$, we have

$$\left.\begin{aligned} &e^\Theta = Z + X_\alpha c^\dagger_\alpha - \bar{X}_\alpha c_\alpha,\quad \bar{X}_\alpha X_\alpha + Z^2 = 1,\\ &Z = \cos\theta,\quad X_\alpha = \frac{\theta_\alpha}{\theta}\sin\theta,\quad \theta^2 = \bar{\theta}_\alpha\theta_\alpha. \end{aligned}\right\} \tag{2.13}$$

From (2.1), (2.12) and (2.13), we obtain

$$\left.\begin{aligned} &e^\Theta(c_\alpha, c^\dagger_\alpha, \frac{1}{\sqrt{2}})(-1)^n e^{-\Theta} = (c_\beta, c^\dagger_\beta, \frac{1}{\sqrt{2}})(-1)^n G_X,\\ &G_X \stackrel{\mathrm{def}}{=} \begin{bmatrix} \delta_{\beta\alpha} - \bar{X}_\beta X_\alpha & \bar{X}_\beta\bar{X}_\alpha & -\sqrt{2}Z\bar{X}_\beta \\ X_\beta X_\alpha & \delta_{\beta\alpha} - X_\beta\bar{X}_\alpha & \sqrt{2}ZX_\beta \\ \sqrt{2}ZX_\alpha & -\sqrt{2}Z\bar{X}_\alpha & 2Z^2-1 \end{bmatrix}. \end{aligned}\right\} \tag{2.14}$$

Let G be the $(2N+1)\times(2N+1)$ matrix defined by

$$G \stackrel{\mathrm{def}}{=} G_X\begin{bmatrix} a & \bar{b} & 0 \\ b & \bar{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\bar{X}Y & \bar{b}+\bar{X}\bar{Y} & -\sqrt{2}Z\bar{X} \\ b+XY & \bar{a}-X\bar{Y} & \sqrt{2}ZX \\ \sqrt{2}ZY & -\sqrt{2}Z\bar{Y} & 2Z^2-1 \end{bmatrix},\ \left.\begin{aligned} X_\alpha&=\bar{a}^\alpha{}_\beta Y_\beta - b_{\alpha\beta}\bar{Y}_\beta,\\ Y_\alpha&=X_\beta a^\beta{}_\alpha - \bar{X}_\beta b_{\beta\alpha},\\ \bar{Y}_\alpha Y_\alpha &+ Z^2 = 1, \end{aligned}\right\} \tag{2.15}$$

where X and Y are the column vector and the row vector, respectively. The $SO(2N+1)$ canonical transformation $U(G)$ is generated by the fermion $SO(2N+1)$ Lie operators. The $U(G)$ is an extension of the generalized Bogoliubov transformation $U(g)$ [7] to a nonlinear transformation and is specified by the $SO(2N+1)$ matrix G. We identify this G with the argument G of $U(G)$. Then $U(G)=U(G_X)U(g)$ and $U(G_X)=\exp(\Theta)$.

From (2.6), (2.14) and (2.15) and the commutability of $U(g)$ with $(-1)^n$, we obtain

$$U(G)(c_\alpha, c_\alpha^\dagger, \frac{1}{\sqrt{2}})(-1)^n U^\dagger(G) = (c_\beta, c_\beta^\dagger, \frac{1}{\sqrt{2}})(-1)^n \begin{bmatrix} A_{\beta\alpha} & \bar{B}_{\beta\alpha} & -\frac{\bar{x}_\beta}{\sqrt{2}} \\ B_{\beta\alpha} & \bar{A}_{\beta\alpha} & \frac{x_\beta}{\sqrt{2}} \\ \frac{y_\alpha}{\sqrt{2}} & -\frac{\bar{y}_\alpha}{\sqrt{2}} & z \end{bmatrix}, \tag{2.16}$$

where

$$\left.\begin{aligned} A_{\alpha\beta} &= a_{\alpha\beta} - \bar{X}_\alpha Y_\beta = a_{\alpha\beta} - \frac{\bar{x}_\alpha y_\beta}{2(1+z)}, \\ B_{\alpha\beta} &= b_{\alpha\beta} + X_\alpha Y_\beta = b_{\alpha\beta} + \frac{x_\alpha y_\beta}{2(1+z)}, \\ x_\alpha &= 2ZX_\alpha,\ \ y_\alpha = 2ZY_\alpha,\ \ z = 2Z^2 - 1. \end{aligned}\right\} \tag{2.17}$$

By using the relation $U(G)(c, c^\dagger, \frac{1}{\sqrt{2}})U^\dagger(G) = U(G)(c, c^\dagger, \frac{1}{\sqrt{2}})U^\dagger(G)(z+\rho)(-1)^n$ and the third column equation of (2.16), Eq. (2.16) can be written as

$$U(G)(c, c^\dagger, \frac{1}{\sqrt{2}})U^\dagger(G) = (c, c^\dagger, \frac{1}{\sqrt{2}})(z-\rho)G, \tag{2.18}$$

$$G \overset{\text{def}}{=} \begin{bmatrix} A & \bar{B} & -\frac{\bar{x}}{\sqrt{2}} \\ B & \bar{A} & \frac{x}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} & -\frac{\bar{y}}{\sqrt{2}} & z \end{bmatrix}, \qquad G^\dagger G = GG^\dagger = 1_{2N+1}, \qquad \det G = 1, \tag{2.19}$$

$$U(G)U(G') = U(GG'), \quad U(G^{-1}) = U^{-1}(G) = U^\dagger(G), \quad U(1_{2N+1}) = \mathbb{I}_G, \tag{2.20}$$

where $(c,\ c^\dagger,\ \frac{1}{\sqrt{2}})$ is a $(2N{+}1)$-dimensional row vector $((c_\alpha),\ (c_\alpha^\dagger),\ \frac{1}{\sqrt{2}})$ and $A = (A^\alpha{}_\beta)$ and $B = (B_{\alpha\beta})$ are $N{\times}N$ matrices. The $U(G)$ is a nonlinear transformation with a q-number gauge factor $z - \rho$ where $\rho = x_\alpha c_\alpha^\dagger - \bar{x}_\alpha c_\alpha$ and $\rho^2 = -\bar{x}_\alpha x_\alpha = z^2 - 1$ [14]. The matrix G is a matrix belonging to the $SO(2N+1)$ group. It can be transformed to a real $(2N+1)$-dimensional orthogonal matrix by the transformation

$$O = VGV^{-1}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1_N & \frac{1}{\sqrt{2}} \cdot 1_N & 0 \\ -\frac{i}{\sqrt{2}} \cdot 1_N & \frac{i}{\sqrt{2}} \cdot 1_N & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.21}$$

When $z = 1$, the G becomes an $SO(2N)$ matrix g. The $SO(2N+1)$ WF $|\,G\!> = U(G)|\,0\!>$ is expressed as [17, 18]

$$\left.\begin{aligned} |\,G\!> &= <\!0\,|U(G)|\,0\!>(1 + r_\alpha c_\alpha^\dagger)\exp(\frac{1}{2} \cdot q_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger)|\,0\!>, \\ r_\alpha &= \frac{1}{1+z}(x_\alpha + q_{\alpha\beta}\bar{x}_\beta), \end{aligned}\right\} \tag{2.22}$$

$$<\!0\,|\,U(G)\,|\,0\!> = \bar{\Phi}_{00}(G) = \sqrt{\frac{1+z}{2}}\left[\det(1_N + q^\dagger q)\right]^{-\frac{1}{4}} e^{i\frac{\tau}{2}}. \tag{2.23}$$

3 Embedding into an SO(2N+2) group

Following Fukutome [18], we define the projection operators P_+ and P_- onto the sub-spaces of even and odd fermion numbers, respectively, by

$$P_\pm \stackrel{\text{def}}{=} \frac{1}{2}(1 \pm (-1)^n), \quad P_\pm^2 = P_\pm, \quad P_+P_- = 0, \tag{3.1}$$

and define the following operators with the supurious index 0:

$$\left.\begin{aligned}
&E^0_{\ 0} \stackrel{\text{def}}{=} -\frac{1}{2}(-1)^n = \frac{1}{2}(P_- - P_+),\\
&E^\alpha_{\ 0} \stackrel{\text{def}}{=} c^\dagger_\alpha P_- = P_+ c^\dagger_\alpha, \quad E^0_{\ \alpha} \stackrel{\text{def}}{=} c_\alpha P_+ = P_- c_\alpha,\\
&E^{\alpha 0} \stackrel{\text{def}}{=} -c^\dagger_\alpha P_+ = -P_- c^\dagger_\alpha, \quad E^{0\alpha} \stackrel{\text{def}}{=} -E^{\alpha 0},\\
&E_{\alpha 0} \stackrel{\text{def}}{=} c_\alpha P_- = P_+ c_\alpha, \quad E_{0\alpha} \stackrel{\text{def}}{=} -E_{\alpha 0}.
\end{aligned}\right\} \tag{3.2}$$

The annihilation-creation operators can be expressed in terms of the operators (3.2) as

$$c_\alpha = E_{\alpha 0} + E^0_{\ \alpha}, \quad c^\dagger_\alpha = -E^{\alpha 0} + E^\alpha_{\ 0}. \tag{3.3}$$

We introduce the indices $p, q, \cdots$ running over $N+1$ values $0, 1, \cdots, N$. Then the operators of (2.2) and (3.2) can be denoted in a unified manner as $E^p_{\ q}$, E_{pq} and E^{pq}. They satisfy

$$\left.\begin{aligned}
&E^{p\dagger}_{\ q} = E^q_{\ p}, \quad E^{pq} = E^\dagger_{qp}, \quad E_{pq} = -E_{qp}, \quad (p, q = 0, 1, \cdots, N)\\
&[E^p_{\ q}, E^r_{\ s}] = \delta_{qr} E^p_{\ s} - \delta_{ps} E^r_{\ q}, \quad (U(N+1) \text{ algebra})\\
&\quad \left.\begin{aligned}
&[E^p_{\ q}, E_{rs}] = \delta_{ps} E_{qr} - \delta_{pr} E_{qs},\\
&[E^{pq}, E_{rs}] = \delta_{ps} E^q_{\ r} + \delta_{qr} E^p_{\ s} - \delta_{pr} E^q_{\ s} - \delta_{qs} E^p_{\ r},\\
&[E_{pq}, E_{rs}] = 0.
\end{aligned}\right]
\end{aligned}\right\} \tag{3.4}$$

The above commutation relations in (3.4) are of the same form as (2.3) and (2.4).

Instead of (3.2), it is possible to employ the operators

$$\tilde{E}^0_{\ 0} = \frac{1}{2}(-1)^n = \frac{1}{2}(P_+ - P_-), \quad \tilde{E}^\alpha_{\ 0} = c^\dagger_\alpha P_+, \quad \tilde{E}^0_{\ \alpha} = c_\alpha P_-. \tag{3.5}$$

Denoting $E^\alpha_{\ \beta} \equiv \tilde{E}^\alpha_{\ \beta}$, it is shown that the operators $\tilde{E}^p_{\ q}$, $p, q = 0, 1, \cdots, N$, satisfy

$$\tilde{E}^{p\dagger}_{\ q} = \tilde{E}^q_{\ p}, \quad [\tilde{E}^p_{\ q}, \tilde{E}^r_{\ s}] = \delta_{qr} \tilde{E}^p_{\ s} - \delta_{ps} \tilde{E}^r_{\ q}. \quad (\tilde{U}(N+1) \text{ algebra}) \tag{3.6}$$

The Lie algebra $\tilde{U}(N+1)$ is a $U(N+1)$ Lie algebra but it is not unitarily equivalent to $U(N+1)$.

Two Clifford algebras C_{2N} and C_{2N+1} can be constructed from the fermion operators:

$$M_\alpha = c_\alpha + c^\dagger_\alpha, \quad M_{\alpha+N} = i(c_\alpha - c^\dagger_\alpha), \tag{3.7}$$

$$\bar{M}_\alpha = (c_\alpha - c^\dagger_\alpha)(-1)^n, \quad \bar{M}_{\alpha+N} = i(c_\alpha + c^\dagger_\alpha)(-1)^n, \quad \bar{M}_0 = (-1)^n. \tag{3.8}$$

These operators satisfy

$$\{M_i,\ M_j\} = 2\delta_{ij}, \quad (i,\ j = 1,\ \cdots,\ 2N) \tag{3.9}$$

$$\{\bar{M}_i,\ \bar{M}_j\} = 2\delta_{ij}. \quad (i,\ j = 0,\ 1,\ \cdots,\ 2N) \tag{3.10}$$

$C_{2N} = \{M_i; i = 1,\ \cdots,\ 2N\}$ and $C_{2N+1} = \{\bar{M}_i; i = 0,\ 1,\ \cdots,\ 2N\}$ are the Clifford algebras of $2N$ and $2N+1$ dimensions, respectively [19, 20]. They provide the bases to characterize the canonical transformations generated by the $SO(2N)$ and $SO(2N+1)$ Lie algebras.

The $SO(2N+1)$ group is embedded into an $SO(2N+2)$ group. The embedding leads us to an unified formulation of the $SO(2N+1)$ regular representation in which paired and unpaired modes are treated in an equal way. Define $(N{+}1)\times(N{+}1)$ matrices $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A} = \begin{bmatrix} A & -\dfrac{\bar{x}}{2} \\ \dfrac{y}{2} & \dfrac{1+z}{2} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & \dfrac{x}{2} \\ -\dfrac{y}{2} & \dfrac{1-z}{2} \end{bmatrix}, \quad y = x^{\mathrm{T}}a - x^{\dagger}b. \tag{3.11}$$

Imposing the ortho-normalization of the G, matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy the ortho-normalization condition and then form an $SO(2N+2)$ matrix $\mathcal{G}$ represented as [17]

$$\mathcal{G} = \begin{bmatrix} \mathcal{A} & \bar{\mathcal{B}} \\ \mathcal{B} & \bar{\mathcal{A}} \end{bmatrix}, \quad \mathcal{G}^{\dagger}\mathcal{G} = \mathcal{G}\mathcal{G}^{\dagger} = 1_{2N+2}, \tag{3.12}$$

which means the ortho-normalization conditions of the $N+1$-dimensional HB amplitudes

$$\left.\begin{array}{l} \mathcal{A}^{\dagger}\mathcal{A} + \mathcal{B}^{\dagger}\mathcal{B} = 1_{N+1}, \quad \mathcal{A}^{\mathrm{T}}\mathcal{B} + \mathcal{B}^{\mathrm{T}}\mathcal{A} = 0, \\[2ex] \mathcal{A}\mathcal{A}^{\dagger} + \bar{\mathcal{B}}\mathcal{B}^{\mathrm{T}} = 1_{N+1}, \quad \bar{\mathcal{A}}\mathcal{B}^{\mathrm{T}} + \mathcal{B}\mathcal{A}^{\dagger} = 0. \end{array}\right\} \tag{3.13}$$

The matrix $\mathcal{G}$ satisfies $\det\mathcal{G} = 1$ as is proved easily below

$$\det\mathcal{G} = \det\left(\mathcal{A} - \bar{\mathcal{B}}\bar{\mathcal{A}}^{-1}\mathcal{B}\right)\det\bar{\mathcal{A}} = \det\left(\mathcal{A}\mathcal{A}^{\dagger} - \bar{\mathcal{B}}\bar{\mathcal{A}}^{-1}\mathcal{B}\mathcal{A}^{\dagger}\right) = 1. \tag{3.14}$$

By using (2.17) and (2.15), the matrices $\mathcal{A}$ and $\mathcal{B}$ can be decomposed as

$$\mathcal{A} = \begin{bmatrix} 1_N - \dfrac{\bar{x}r^{\mathrm{T}}}{2} & -\dfrac{\bar{x}}{2} \\ \dfrac{(1+z)r^{\mathrm{T}}}{2} & \dfrac{1+z}{2} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1_N + \dfrac{xr^{\mathrm{T}}q^{-1}}{2} & \dfrac{x}{2} \\ -\dfrac{(1+z)r^{\mathrm{T}}q^{-1}}{2} & \dfrac{1-z}{2} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.15}$$

from which we get the inverse of $\mathcal{A}$, $\mathcal{A}^{-1}$, as

$$\mathcal{A}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_N & \dfrac{\bar{x}}{1+z} \\ -r^{\mathrm{T}} & 1 \end{bmatrix}. \tag{3.16}$$

From (3.15) and (3.16), we obtain a $\frac{SO(2N+2)}{U(N+1)}$ coset variable with the $N{+}1$-th component as

$$\mathcal{Q} = \mathcal{B}\mathcal{A}^{-1} = \begin{bmatrix} q & r \\ -r^{\mathrm{T}} & 0 \end{bmatrix} = -\mathcal{Q}^{\mathrm{T}}, \tag{3.17}$$

from which the $SO(2N+1)$ variables $q_{\alpha\beta}$ and r_α are shown to be just the independent variables of the $\frac{SO(2N+2)}{U(N+1)}$ coset space. The paired mode $q_{\alpha\beta}$ and unpaired mode r_α variables in the $SO(2N+1)$ algebra are unified as the paired variables in the $SO(2N+2)$ algebra [17]. We denote the $(N+1)$-dimension of the matrix Q by the index 0 and use the indices $p,\ q,\ \cdots$ running over 0 and $\alpha,\ \beta,\ \cdots$.

4 σ-model on the SO(2N+2)/U(N+1) coset manifold

Let us introduce a $(2N+2)\times(N+1)$ isometric matrix $\mathcal{U}$ by

$$\mathcal{U}^{\mathrm{T}} = \left[\ \mathcal{B}^{\mathrm{T}},\ \mathcal{A}^{\mathrm{T}}\ \right]. \tag{4.1}$$

If one uses the matrix elements of $\mathcal{U}$ and $\mathcal{U}^\dagger$ as the co-ordinates on the manifold $SO(2N+2)$, a real line element can be defined by a hermitian metric tensor on the manifold. Under the transformation $\mathcal{U}\to\mathcal{V}\mathcal{U}$ the metric is invariant. Then the metric tensor defined on the manifold may become singular, due to the fact that one uses too many co-ordinates.

According to Zumino [1], if $\mathcal{A}$ is non-singular, we have relations governing $\mathcal{U}^\dagger\mathcal{U}$ as

$$\left.\begin{aligned}
&\mathcal{U}^\dagger\mathcal{U} = \mathcal{A}^\dagger\mathcal{A}+\mathcal{B}^\dagger\mathcal{B} = \mathcal{A}^\dagger\left\{1_{N+1}+\left(\mathcal{B}\mathcal{A}^{-1}\right)^\dagger\left(\mathcal{B}\mathcal{A}^{-1}\right)\right\}\mathcal{A} = \mathcal{A}^\dagger\left(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q}\right)\mathcal{A},\\
&\ln\det\mathcal{U}^\dagger\mathcal{U} = \ln\det\left(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q}\right)+\ln\det\mathcal{A}+\ln\det\mathcal{A}^\dagger,
\end{aligned}\right\} \tag{4.2}$$

where we have used the $\frac{SO(2N+2)}{U(N+1)}$ coset variable $\mathcal{Q}$ (3.17). If we take the matrix elements of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ as the co-ordinates on the $\frac{SO(2N+2)}{U(N+1)}$ coset manifold, the real line element can be well defined by a hermitian metric tensor on the coset manifold as

$$ds^2 = \mathcal{G}_{pq\ \underline{rs}}d\mathcal{Q}^{pq}d\bar{\mathcal{Q}}^{\underline{rs}}\ \left(\mathcal{Q}^{pq}=\mathcal{Q}_{pq}\text{ and }\bar{\mathcal{Q}}^{\underline{rs}}=\bar{\mathcal{Q}}_{\underline{rs}};\ \mathcal{G}_{pq\ \underline{rs}}=\mathcal{G}_{\underline{rs}\ pq}\right). \tag{4.3}$$

We also use the indices $\underline{r},\ \underline{s},\ \cdots$ running over 0 and $\alpha,\ \beta,\ \cdots$. The condition that the manifold under consideration is a Kähler manifold is that its complex structure is covariantly constant relative to the Riemann connection:

$$\mathcal{G}_{pq\ \underline{rs}\ ,tu} \stackrel{\text{def}}{=} \frac{\partial\mathcal{G}_{pq\ \underline{rs}}}{\partial\mathcal{Q}^{tu}} = \mathcal{G}_{tu\ \underline{rs}\ ,pq},\quad \mathcal{G}_{pq\ \underline{rs}\ ,\underline{tu}} \stackrel{\text{def}}{=} \frac{\partial\mathcal{G}_{pq\ \underline{rs}}}{\partial\bar{\mathcal{Q}}^{\underline{tu}}} = \mathcal{G}_{pq\ \underline{tu}\ ,\underline{rs}}, \tag{4.4}$$

and that it has vanishing torsions. Then, the hermitian metric tensor $\mathcal{G}_{pq\ \underline{rs}}$ can be locally given through a real scalar function, the Kähler potential, which takes the well-known form

$$\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q}) = \ln\det\left(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q}\right), \tag{4.5}$$

and the explicit expression for the components of the metric tensor is given as

$$\begin{aligned}
\mathcal{G}_{pq\ \underline{rs}} = \frac{\partial^2\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q})}{\partial\mathcal{Q}^{pq}\partial\bar{\mathcal{Q}}^{\underline{rs}}} &= \left\{\left(1_{N+1}+\mathcal{Q}\mathcal{Q}^\dagger\right)^{-1}\right\}_{sp}\left\{\left(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q}\right)^{-1}\right\}_{qr}\\
&\quad -(r\leftrightarrow s)-(p\leftrightarrow q)+(p\leftrightarrow q,\ r\leftrightarrow s).
\end{aligned} \tag{4.6}$$

Notice that the above function does not determine the Kähler potential $\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q})$ uniquely since the metric tensor $\mathcal{G}_{pq\underline{rs}}$ is invariant under transformations of the Kähler potential,

$$\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q}) \to \mathcal{K}'(\mathcal{Q}^\dagger,\ \mathcal{Q}) = \mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q})+\mathcal{F}(\mathcal{Q})+\bar{\mathcal{F}}(\bar{\mathcal{Q}}). \tag{4.7}$$

$\mathcal{F}(\mathcal{Q})$ and $\bar{\mathcal{F}}(\bar{\mathcal{Q}})$ are analytic functions of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, respectively. In the case of the Kähler metric tensor, we have only the components of the metric connections with unmixed indices

$$\Gamma^{\ \ tu}_{pq\ rs} = \mathcal{G}^{\underline{vw}\ tu}\mathcal{G}_{pq\ \underline{vw}\ ,rs},\quad \bar{\Gamma}^{\ \ \underline{tu}}_{\underline{pq}\ \underline{rs}} = \mathcal{G}^{\underline{tu}\ vw}\mathcal{G}_{vw\ \underline{rs}\ ,\underline{pq}},\quad \mathcal{G}^{\underline{vw}\ tu} \stackrel{\text{def}}{=} \left(\mathcal{G}^{-1}\right)_{\underline{vw}\ tu}, \tag{4.8}$$

and only the components of the curvatures

$$\left.\begin{aligned}
&\boldsymbol{R}_{pq\ \underline{rs}\ tu\ \underline{vw}} = \mathcal{G}_{vw\ \underline{vw}}\Gamma^{\ \ vw}_{pq\ tu\ ,\underline{rs}} = \mathcal{G}_{pq\ \underline{vw}\ ,tu\ \underline{rs}} - \mathcal{G}_{t'u'\ \underline{v'w'}}\Gamma^{\ \ t'u'}_{pq\ tu}\bar{\Gamma}^{\ \ \underline{v'w'}}_{\underline{rs}\ \underline{vw}},\\
&\boldsymbol{R}_{\underline{rs}\ pq\ \underline{vw}\ tu} = \mathcal{G}_{tu\ \underline{tu}}\bar{\Gamma}^{\ \ \underline{tu}}_{\underline{rs}\ \underline{vw}\ ,pq} = \boldsymbol{R}_{pq\ \underline{rs}\ tu\ \underline{vw}}.
\end{aligned}\right\} \tag{4.9}$$

In two- or four-dimensional space-time, the simplest representation of $\mathcal{N}=1$ supersymmetry is a scalar multiplet $\phi = \{\mathcal{Q}, \psi_L, H\}$ where $\mathcal{Q}$ and H are complex scalars and $\psi_L \equiv \frac{1}{2}(1+\gamma_5)\psi$ is a left-handed chiral spinor defined through a Majorana spinor. In superspace language the multiplet is written as a chiral superfield:

$$\phi = \mathcal{Q} + \bar{\theta}_R\psi_L + \bar{\theta}_R\theta_L H. \tag{4.10}$$

The most general theory of the supersymmetric σ-model can be constructed from the $[N]$ scalar multiplets $\phi^{[\alpha]} = \{\mathcal{Q}^{[\alpha]}, \psi_L^{[\alpha]}, H^{[\alpha]}\}([\alpha]=1,\ \cdots,\ [N])$. The supersymmetry transformations are given by

$$\delta\mathcal{Q}^{[\alpha]} = \bar{\varepsilon}_R\psi_L^{[\alpha]}, \quad \delta\psi_L^{[\alpha]} = \frac{1}{2}(\partial\!\!\!/\mathcal{Q}^{[\alpha]}\varepsilon_R + H^{[\alpha]}\varepsilon_L), \quad \delta H^{[\alpha]} = \bar{\varepsilon}_L\partial\!\!\!/\psi_L^{[\alpha]}, \tag{4.11}$$

where ε is the Majorana spinor parameter.

Let the Kähler manifold be the $\frac{SO(2N+2)}{U(N+1)}$ coset manifold and redenote the complex scalar fields $\mathcal{Q}_{pq}$ as $\mathcal{Q}^{[\alpha]}([\alpha]=1,\cdots,\ \frac{N(N+1)}{2}\ (=[N]))$ and suppose that the spinors $\psi_L^{[\alpha]}$ and $\bar{\psi}_L^{[\alpha]}$ span the fibres. Following Zumino [1] and van Holten et al. [13], the Lagrangian of a supersymmetric σ-model can be written completely in terms of co-ordinates on the fibre bundle in the following form:

$$\begin{aligned}
\mathcal{L}_{\text{chiral}} = &-\mathcal{G}_{[\alpha][\underline{\beta}]}\left(\partial_\mu\bar{\mathcal{Q}}^{[\underline{\beta}]}\partial_\mu\mathcal{Q}^{[\alpha]} + \bar{\psi}_L^{[\underline{\beta}]}\overleftrightarrow{\boldsymbol{D}\!\!\!/}\psi_L^{[\alpha]}\right) + W_{;[\alpha][\beta]}\bar{\psi}_R^{[\beta]}\psi_L^{[\alpha]} + \bar{W}_{;[\underline{\alpha}][\underline{\beta}]}\bar{\psi}_L^{[\underline{\beta}]}\psi_R^{[\underline{\alpha}]}\\
&-\mathcal{G}_{[\alpha][\underline{\alpha}]}\bar{W}_{;[\underline{\alpha}]}W_{;[\alpha]} + \frac{1}{2}\boldsymbol{R}_{[\alpha][\underline{\beta}][\gamma][\underline{\delta}]}\bar{\psi}_L^{[\underline{\beta}]}\gamma_\mu\psi_L^{[\alpha]}\bar{\psi}_L^{[\underline{\delta}]}\gamma_\mu\psi_L^{[\gamma]},
\end{aligned} \tag{4.12}$$

and the Kähler covariant derivative is defined as $\boldsymbol{D}_\mu\psi_L^{[\alpha]} \stackrel{\text{def}}{=} \partial_\mu\psi_L^{[\alpha]} + \Gamma^{\ \ [\alpha]}_{[\beta][\gamma]}\psi_L^{[\beta]}\partial_\mu\mathcal{Q}^{[\gamma]}$. The curvature tensors $\boldsymbol{R}$ are given by (4.9). This form of the Lagrangian has also been derived by Higashijima and Nitta with the use of the Kähler normal co-ordinate expansion of the Lagrangian $\mathcal{L} = \int d^4\theta\mathcal{K}(\phi^\dagger,\ \phi)$ [21]. The Lagrangian $\mathcal{L}_{\text{chiral}}$ (4.12) is manifestly a scalar under the general co-ordinate transformations $\mathcal{Q}^{[\alpha]} \to \mathcal{Q}'^{[\alpha]}$ on the manifold, provided that the fermion transforms as a vector and the superpotential does as a scalar:

$$\psi'^{[\alpha]}_L = \frac{\partial\mathcal{Q}'^{[\alpha]}}{\partial\mathcal{Q}^{[\beta]}}\psi_L^{[\beta]}, \quad W'(\mathcal{Q}') = W(\mathcal{Q}). \tag{4.13}$$

The auxiliary fields $H^{[\alpha]}$ are eliminated through their field equations

$$H^{[\alpha]} = \Gamma^{\ \ [\alpha]}_{[\beta][\gamma]}\bar{\psi}_R^{[\beta]}\psi_L^{[\gamma]} + \mathcal{G}^{[\alpha][\underline{\beta}]}\bar{W}_{,[\underline{\beta}]}. \tag{4.14}$$

The right-handed chiral spinor ψ_R is defined as $\psi_R = C\bar{\psi}_L^{\mathrm{T}}$ and C is the charge conjugation. The comma $,[\alpha]$ $(,[\underline{\alpha}])$ denotes a derivative with respect to $\mathcal{Q}^{[\alpha]}$ $(\bar{\mathcal{Q}}^{[\underline{\alpha}]})$, while the semicolon denotes a covariant derivative using the affine connection defined by the $\Gamma^{\ \ [\alpha]}_{[\beta][\gamma]}$:

$$W_{;[\alpha]} = W_{,[\alpha]} = \frac{\partial W}{\partial\mathcal{Q}^{[\alpha]}}, \quad W_{;[\alpha][\beta]} = \frac{\partial W_{;[\alpha]}}{\partial\mathcal{Q}^{[\beta]}} - \Gamma^{\ \ [\gamma]}_{[\beta][\alpha]}W_{;[\gamma]}. \tag{4.15}$$

5 Expression for SO(2N+2)/U(N+1) Killing potential

Let us consider an $SO(2N+2)$ infinitesimal left transformation of an $SO(2N+2)$ matrix $\mathcal{G}$ to $\mathcal{G}'$, $\mathcal{G}' = (1_{2N+2} + \delta\mathcal{G})\mathcal{G}$, by using the first equation of (A.3):

$$\mathcal{G}' = \left[\begin{array}{cc} 1_{N+1} + \delta\mathcal{A} & \delta\bar{\mathcal{B}} \\ \delta\mathcal{B} & 1_{N+1} + \delta\bar{\mathcal{A}} \end{array}\right] \mathcal{G} = \left[\begin{array}{cc} \mathcal{A} + \delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\mathcal{B} & \bar{\mathcal{B}} + \delta\mathcal{A}\bar{\mathcal{B}} + \delta\bar{\mathcal{B}}\bar{\mathcal{A}} \\ \mathcal{B} + \delta\bar{\mathcal{A}}\mathcal{B} + \delta\mathcal{B}\mathcal{A} & \bar{\mathcal{A}} + \delta\bar{\mathcal{A}}\bar{\mathcal{A}} + \delta\mathcal{B}\bar{\mathcal{B}} \end{array}\right]. \tag{5.1}$$

Let us define a $\frac{SO(2N+2)}{U(N+1)}$ coset variable $\mathcal{Q}'(=\mathcal{B}'\mathcal{A}'^{-1})$ in the $\mathcal{G}'$ frame. With the aid of (5.1), the $\mathcal{Q}'$ is calculated infinitesimally as

$$\begin{aligned} \mathcal{Q}' = \mathcal{B}'\mathcal{A}'^{-1} &= \left(\mathcal{B} + \delta\bar{\mathcal{A}}\mathcal{B} + \delta\mathcal{B}\mathcal{A}\right)\left(\mathcal{A} + \delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\mathcal{B}\right)^{-1} \\ &= \mathcal{Q} + \delta\mathcal{B} - \mathcal{Q}\delta\mathcal{A} + \delta\bar{\mathcal{A}}\mathcal{Q} - \mathcal{Q}\delta\bar{\mathcal{B}}\mathcal{Q}. \end{aligned} \tag{5.2}$$

The Kähler metrics admit a set of holomorphic isometries, the Killing vectors, $\mathcal{R}^{i[\alpha]}(\mathcal{Q})$ and $\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}})$ $(i\!=\!1,\cdots,\dim\mathcal{G})$, which are the solution of the Killing equation

$$\mathcal{R}^i{}_{[\underline{\beta}]}(\mathcal{Q}),{}_{[\alpha]} + \bar{\mathcal{R}}^i{}_{[\alpha]}(\mathcal{Q}),{}_{[\underline{\beta}]} = 0, \quad \mathcal{R}^i{}_{[\underline{\beta}]}(\mathcal{Q}) = \mathcal{G}_{[\alpha][\underline{[\beta]}]}\mathcal{R}^{i[\alpha]}(\mathcal{Q}). \tag{5.3}$$

These isometries define infinitesimal symmetry transformations and are described geometrically by the above Killing vectors which are the generators of infinitesimal co-ordinate transformations keeping the metric invariant: $\delta\mathcal{Q}\!=\!\mathcal{Q}'\!-\!\mathcal{Q}\!=\!\mathcal{R}(\mathcal{Q})$ and $\delta\bar{\mathcal{Q}}\!=\!\bar{\mathcal{R}}(\bar{\mathcal{Q}})$ such that $\mathcal{G}'(\mathcal{Q},\bar{\mathcal{Q}})\!=\!\mathcal{G}(\mathcal{Q},\bar{\mathcal{Q}})$. The Killing equation (5.3) is the necessary and sufficient condition for an infinitesimal co-ordinate transformation

$$\delta\mathcal{Q}^{[\alpha]} = \left(\delta\mathcal{B} - \delta\mathcal{A}^{\mathrm{T}}\mathcal{Q} - \mathcal{Q}\delta\mathcal{A} + \mathcal{Q}\delta\mathcal{B}^{\dagger}\mathcal{Q}\right)^{[\alpha]} = \xi_i\mathcal{R}^{i[\alpha]}(\mathcal{Q}), \quad \delta\bar{\mathcal{Q}}^{[\underline{\alpha}]} = \xi_i\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}}), \tag{5.4}$$

where ξ_i are the infinitesimal and global group parameters. Due to the Killing equation, the Killing vectors $\mathcal{R}^{i[\alpha]}(\mathcal{Q})$ and $\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}})$ can be written locally as the gradient of some real scalar function, the Killing potentials $\mathcal{M}^i(\mathcal{Q},\bar{\mathcal{Q}})$ such that

$$\mathcal{R}^i{}_{[\underline{\alpha}]}(\mathcal{Q}) = -i\mathcal{M}^i{}_{,[\underline{\alpha}]}, \quad \bar{\mathcal{R}}^i{}_{[\alpha]}(\bar{\mathcal{Q}}) = i\mathcal{M}^i{}_{,[\alpha]}. \tag{5.5}$$

According to van Holten et al. [2] and using the infinitesimal $SO(2N+2)$ matrix $\delta\mathcal{G}$ given by the first equation of (A.3), the Killing potential $\mathcal{M}_\sigma$ can be written for the coset $\frac{SO(2N+2)}{U(N+1)}$ as

$$\left.\begin{aligned} &\mathcal{M}_\sigma\left(\delta\mathcal{A},\delta\mathcal{B},\delta\mathcal{B}^{\dagger}\right) = \mathrm{Tr}\left(\delta\mathcal{G}\widetilde{\mathcal{M}}_\sigma\right) = \mathrm{tr}\left(\delta\mathcal{A}\mathcal{M}_{\sigma\delta\mathcal{A}} + \delta\mathcal{B}\mathcal{M}_{\sigma\delta\mathcal{B}^{\dagger}} + \delta\mathcal{B}^{\dagger}\mathcal{M}_{\sigma\delta\mathcal{B}}\right), \\ &\widetilde{\mathcal{M}}_\sigma \equiv \left[\begin{array}{cc} \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} & \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^{\dagger}} \\ -\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}} & -\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}} \end{array}\right], \quad \begin{aligned} &\mathcal{M}_{\sigma\delta\mathcal{A}} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} + \left(\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}}\right)^{\mathrm{T}}, \\ &\mathcal{M}_{\sigma\delta\mathcal{B}} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}}, \quad \mathcal{M}_{\sigma\delta\mathcal{B}^{\dagger}} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^{\dagger}}, \end{aligned} \end{aligned}\right\} \tag{5.6}$$

where the trace Tr is taken over the $(2N+2)\!\times\!(2N+2)$ matrices, while the trace tr is taken over the $(N+1)\times(N+1)$ matrices. Let us introduce the $(N+1)$-dimensional matrices $\mathcal{R}(\mathcal{Q};\delta\mathcal{G})$, $\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})$ and $\mathcal{X}$ by

$$\left.\begin{aligned} &\mathcal{R}(\mathcal{Q};\delta\mathcal{G}) = \delta\mathcal{B} - \delta\mathcal{A}^{\mathrm{T}}\mathcal{Q} - \mathcal{Q}\delta\mathcal{A} + \mathcal{Q}\delta\mathcal{B}^{\dagger}\mathcal{Q}, \quad \mathcal{R}_T(\mathcal{Q};\delta\mathcal{G}) = -\delta\mathcal{A}^{\mathrm{T}} + \mathcal{Q}\delta\mathcal{B}^{\dagger}, \\ &\mathcal{X} = (1_{N+1} + \mathcal{Q}\mathcal{Q}^{\dagger})^{-1} = \mathcal{X}^{\dagger}. \end{aligned}\right\} \tag{5.7}$$

In (5.4), putting $\xi_i = 1$, we have $\delta\mathcal{Q} = \mathcal{R}(\mathcal{Q};\delta\mathcal{G})$ which is just the Killing vector in the coset space $\frac{SO(2N+2)}{U(N+1)}$, and tr of the holomorphic matrix-valued function $\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})$, $\mathrm{tr}\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G}) = \mathcal{F}(\mathcal{Q})$ is a holomorphic Kähler transformation. Then the Killing potential $\mathcal{M}_\sigma$ is given as

$$\left.\begin{aligned} &-i\mathcal{M}_\sigma\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right) = -\mathrm{tr}\Delta\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right), \\ &\Delta\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right) \overset{\mathrm{def}}{=} \mathcal{R}_T(\mathcal{Q};\delta\mathcal{G}) - \mathcal{R}(\mathcal{Q};\delta\mathcal{G})\mathcal{Q}^\dagger\mathcal{X} = \left(\mathcal{Q}\delta\mathcal{A}\mathcal{Q}^\dagger - \delta\mathcal{A}^{\mathrm{T}} - \delta\mathcal{B}\mathcal{Q}^\dagger + \mathcal{Q}\delta\mathcal{B}^\dagger\right)\mathcal{X}. \end{aligned}\right\} \tag{5.8}$$

From (5.6) and (5.8), we obtain

$$-i\mathcal{M}_{\sigma\delta\mathcal{B}} = -\mathcal{X}\mathcal{Q}, \quad -i\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger} = \mathcal{Q}^\dagger\mathcal{X}, \quad -i\mathcal{M}_{\sigma\delta\mathcal{A}} = 1_{N+1} - 2\mathcal{Q}^\dagger\mathcal{X}\mathcal{Q}. \tag{5.9}$$

Using the expression for $\widetilde{\mathcal{M}}_\sigma$, equation (5.9), their components are written in the form

$$-i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}} = -\mathcal{X}\mathcal{Q}, \; -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^\dagger} = \mathcal{Q}^\dagger\mathcal{X}, \; -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} = -\mathcal{Q}^\dagger\mathcal{X}\mathcal{Q}, \; -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}} = 1_{N+1} - \mathcal{Q}\bar{\mathcal{X}}\mathcal{Q}^\dagger = \mathcal{X}. \tag{5.10}$$

It is easily checked that the result (5.9) satisfies the gradient of the real function $\mathcal{M}_\sigma$ (5.5). Of course, putting $r = 0$ in $\mathcal{Q}$ (3.17), the Killing potential $\mathcal{M}_\sigma$ in the $\frac{SO(2N+2)}{U(N+1)}$ coset space leads to the Killing potential M_σ in the $\frac{SO(2N)}{U(N)}$ coset space obtained by van Holten et al. [2].

To make clear the meaning of the Killing potential, using the $(2N+2)\times(N+1)$ isometric matrix $\mathcal{U}$ ($\mathcal{U}^\dagger\mathcal{U} = 1_{N+1}$), let us introduce the following $(2N+2)\times(2N+2)$ matrix:

$$\mathcal{W} = \mathcal{U}\mathcal{U}^\dagger = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix}, \quad \begin{aligned} \mathcal{R} &= \mathcal{B}\mathcal{B}^\dagger, \\ \mathcal{K} &= \mathcal{B}\mathcal{A}^\dagger, \end{aligned} \tag{5.11}$$

which satisfies the idempotency relation $\mathcal{W}^2 = \mathcal{W}$ and is hermitian on the $SO(2N+2)$ group. $\mathcal{W}$ is a natural extension of the generalized density matrix in the $SO(2N)$ CS rep to the $SO(2N+2)$ CS rep. Since the matrices $\mathcal{A}$ and $\mathcal{B}$ are represented in terms of $\mathcal{Q} = (\mathcal{Q}_{pq})$ as

$$\mathcal{A} = (1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-\frac{1}{2}}\overset{\circ}{\mathcal{U}}, \quad \mathcal{B} = \mathcal{Q}(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-\frac{1}{2}}\overset{\circ}{\mathcal{U}}, \quad \overset{\circ}{\mathcal{U}} \in U(N+1), \tag{5.12}$$

then, we have

$$\mathcal{R} = \mathcal{Q}(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-1}\mathcal{Q}^\dagger = \mathcal{Q}\bar{\chi}\mathcal{Q}^\dagger = 1_{N+1} - \chi, \quad \mathcal{K} = \mathcal{Q}(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-1} = \chi\mathcal{Q} \tag{5.13}$$

Substituting (5.13) into (5.10), the Killing potential $-i\widetilde{\mathcal{M}}_\sigma$ is expressed in terms of the sub-matrices $\mathcal{R}$ and $\mathcal{K}$ of the generalized density matrix (5.11) as

$$-i\widetilde{\mathcal{M}}_\sigma = \begin{bmatrix} -\bar{\mathcal{R}} & -\bar{\mathcal{K}} \\ \mathcal{K} & -(1_{N+1} - \mathcal{R}) \end{bmatrix}, \tag{5.14}$$

from which we finally obtain

$$-i\overline{\widetilde{\mathcal{M}}}_\sigma = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix}. \tag{5.15}$$

To our great surprise, the expression for the Killing potential (5.15) just becomes equivalent with the generalized density matrix (5.11).

The expression for the Killing potential $\mathcal{M}_\sigma$ is described in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variable $\mathcal{Q}_{pq}$ but include an inverse matrix $\mathcal{X}$ given by (5.7). The variable $\mathcal{Q}_{pq}$ has already been expressed in terms of the variables $q_{\alpha\beta}$ and r_α through (3.17). To obtain the concrete expression for the Killing potential in terms of the $SO(2N+1)$ variables $q_{\alpha\beta}$ and r_α, we must also represent the inverse matrix in terms of the variables $q_{\alpha\beta}$ and r_α. Following Ref. [22], after some algebraic manipulations, the inverse matrix $\mathcal{X}$ in (5.7) leads to the form

$$\mathcal{X} = \begin{bmatrix} \mathcal{Q}_{qq^\dagger} & \mathcal{Q}_{qr} \\ \mathcal{Q}^\dagger_{qr} & \mathcal{Q}_{r^\dagger r} \end{bmatrix}, \quad \chi = (1_N + qq^\dagger)^{-1} = \chi^\dagger, \tag{5.16}$$

where each sub-matrix is expressed by the variables q and r as

$$\mathcal{Q}_{qq^\dagger} = \chi - \frac{1+z}{2}\chi(rr^\dagger - q\bar{r}r^{\mathrm{T}}q^\dagger)\chi, \tag{5.17}$$

$$\mathcal{Q}_{q\bar{r}} = \frac{1+z}{2}\chi q\bar{r}, \quad \mathcal{Q}_{r^\dagger r} = \frac{1+z}{2}. \tag{5.18}$$

Then, substituting (3.17) and (5.16) into (5.9) and introducing an auxiliary function $\lambda = rr^\dagger - q\bar{r}r^{\mathrm{T}}q^\dagger = \lambda^\dagger$, we can get the Killing potential $\mathcal{M}_\sigma$ expressed in terms of only q, r and $1+z = 2Z^2$ as,

$$-i\mathcal{M}_{\sigma\delta\mathcal{B}} = \begin{bmatrix} -\chi q + Z^2\left(\chi\lambda\chi q + \chi q\bar{r}r^{\mathrm{T}}\right) & -\chi r + Z^2\chi\lambda\chi r \\ -Z^2\left(r^{\mathrm{T}}q^\dagger\chi q - r^{\mathrm{T}}\right) & -Z^2 r^{\mathrm{T}}q^\dagger\chi r \end{bmatrix}, \tag{5.19}$$

$$-i\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger} = \begin{bmatrix} q^\dagger\chi - Z^2\left(q^\dagger\chi\lambda\chi + \bar{r}r^{\mathrm{T}}q^\dagger\chi\right) & Z^2\left(q^\dagger\chi q\bar{r} - \bar{r}\right) \\ r^\dagger\chi - Z^2 r^\dagger\chi\lambda\chi & Z^2 r^\dagger\chi q\bar{r} \end{bmatrix}, \tag{5.20}$$

$$-i\mathcal{M}_{\sigma\delta\mathcal{A}} =$$

$$\begin{bmatrix} 1_N - 2q^\dagger\chi q + 2Z^2\left(q^\dagger\chi\lambda\chi q + q^\dagger\chi q\bar{r}r^{\mathrm{T}} + \bar{r}r^{\mathrm{T}}q^\dagger\chi q - \bar{r}r^{\mathrm{T}}\right) & -2q^\dagger\chi r + 2Z^2\left(q^\dagger\chi\lambda\chi r + \bar{r}r^{\mathrm{T}}q^\dagger\chi r\right) \\ -2r^\dagger\chi q + 2Z^2\left(r^\dagger\chi\lambda\chi q + r^\dagger\chi q\bar{r}r^{\mathrm{T}}\right) & 1 - 2r^\dagger\chi r + 2Z^2 r^\dagger\chi\lambda\chi r \end{bmatrix}. \tag{5.21}$$

In the above expressions for the Killing potential $\mathcal{M}_\sigma$, each block-matrix is easily verified to satisfy the following identities and relations:

$$r^{\mathrm{T}}q^\dagger\chi r = 0, \quad r^\dagger\chi q\bar{r} = 0, \quad r^\dagger\chi r = \frac{1-Z^2}{Z^2}, \quad r^\dagger\chi\lambda\chi r = \left(\frac{1-Z^2}{Z^2}\right)^2, \tag{5.22}$$

$$1 - 2r^\dagger\chi r + 2Z^2 r^\dagger\chi\lambda\chi r = 2Z^2 - 1, \tag{5.23}$$

$$\chi\lambda\chi r = \frac{1-Z^2}{Z^2}\chi r, \quad r^\dagger\chi\lambda\chi = \frac{1-Z^2}{Z^2}r^\dagger\chi, \quad q^\dagger\chi q = 1_N - \bar{\chi}. \tag{5.24}$$

Using these identities and relations, we get compact forms of the Killing potential $\mathcal{M}_\sigma$ as,

$$-i\mathcal{M}_{\substack{\sigma\delta\mathcal{B} \\ (\sigma\delta\mathcal{B}^\dagger)}} = \left[\begin{array}{cc} \begin{matrix} -\chi q + Z^2\left(\chi r r^\dagger \chi q + \chi q \bar{r} r^{\mathrm{T}} \bar{\chi}\right) \\ \left(q^\dagger \chi - Z^2\left(q^\dagger \chi r r^\dagger \chi + \bar{\chi} \bar{r} r^{\mathrm{T}} q^\dagger \chi\right)\right) \end{matrix} & \begin{matrix} -Z^2 \chi r \\ (-Z^2 \bar{\chi} \bar{r}) \end{matrix} \\ & \\ \begin{matrix} Z^2 r^{\mathrm{T}} \bar{\chi} \\ \left(Z^2 r^\dagger \chi\right) \end{matrix} & \begin{matrix} 0 \\ (0) \end{matrix} \end{array}\right], \tag{5.25}$$

$$-i\mathcal{M}_{\sigma\delta\mathcal{A}} = \left[\begin{array}{cc} 1_N - 2q^\dagger \chi q + 2Z^2\left(q^\dagger \chi r r^\dagger \chi q - \bar{\chi} \bar{r} r^{\mathrm{T}} \bar{\chi}\right) & -2Z^2 q^\dagger \chi r \\ & \\ -2Z^2 r^\dagger \chi q & 2Z^2 - 1 \end{array}\right]. \tag{5.26}$$

Let us introduce the gauge covariant derivatives

$$\left.\begin{aligned} \boldsymbol{D}_\mu \mathcal{Q}^{[\alpha]} &= \partial_\mu \mathcal{Q}^{[\alpha]} - g_i A^i{}_\mu \mathcal{R}^{i[\alpha]}(\mathcal{Q}), \\ \boldsymbol{D}_\mu \psi_L^{[\alpha]} &= \partial_\mu \psi_L^{[\alpha]} - g_i A^i{}_\mu \mathcal{R}^{i[\alpha]}{}_{,[\beta]}(\mathcal{Q}) \psi_L^{[\beta]} + \Gamma^{[\alpha]}_{[\beta][\gamma]} \psi_L^{[\beta]} \partial_\mu \mathcal{Q}^{[\gamma]}, \end{aligned}\right\} \tag{5.27}$$

where $A^i{}_\mu$ are gauge fields corresponding to local symmetries and g_i are coupling constants. They are components of vector multiplets $V^i = (A^i{}_\mu, \lambda^i, D^i)$, with λ^i representing the gauginos and D^i the real auxiliary fields. With the introduction of the gauge fields in Lagrangian (4.12), via the gauge covariant derivatives (5.27), the σ-model is no longer invariant under the supersymmetry transformations. To restore the supersymmetry, it is necessary to add the terms

$$\Delta\mathcal{L}_{\text{chiral}} = 2\mathcal{G}_{[\alpha][\underline{\alpha}]}\left(\mathcal{R}^i{}_{[\underline{\alpha}]}(\mathcal{Q}) \bar{\psi}_L^{[\underline{\alpha}]} \lambda_R^i + \bar{\mathcal{R}}^i{}_{[\alpha]}(\mathcal{Q}) \bar{\lambda}_R^i \psi_L^{[\alpha]}\right) - g_i \mathrm{tr}\left\{D^i(\mathcal{M}^i + \xi^i)\right\}, \tag{5.28}$$

where ξ_i are Fayet-Ilipoulos parameters. Then the full Lagrangian for this model consists of the usual supersymmetry Yang-Mills part and the chiral part

$$\mathcal{L} = -\mathrm{tr}\left\{\frac{1}{4}\mathcal{F}^i_{\mu\nu}\mathcal{F}^i_{\mu\nu} + \frac{1}{2}\bar{\lambda}^i \not{D} \lambda^i - \frac{1}{2}D^i D^i\right\} + \mathcal{L}_{\text{chiral}}(\partial_\mu \to \boldsymbol{D}_\mu) + \Delta\mathcal{L}_{\text{chiral}}. \tag{5.29}$$

Eliminating the auxiliary field D^i by $D^i = -g_i(\mathcal{M}^i + \xi^i)$ (not summed for i), we can get a scalar potential

$$V_{\text{SC}} = -\frac{1}{2} g_i^2 \mathrm{tr}\left\{(\mathcal{M}^i + \xi^i)^2\right\}, \tag{5.30}$$

in which a reduced scalar potential arising from the gauging of $SU(N+1) \times U(1)$ including a Fayet-Ilipoulos term with parameter ξ is of special interest:

$$V_{\text{redSC}} = \frac{g^2_{U(1)}}{2(N+1)}\left(\xi - i\mathcal{M}_Y\right)^2 + \frac{g^2_{SU(N+1)}}{2}\mathrm{tr}\left(-i\mathcal{M}_t\right)^2. \tag{5.31}$$

The new quantities $\mathrm{tr}\,(-i\mathcal{M}_t)^2$ and $-i\mathcal{M}_Y$ in V_{redSC} are defined below and the trace $\mathbf{tr}$, taken over the $N \times N$ matrix, is used. Then we have

$$\left.\begin{aligned} \mathrm{tr}(-i\mathcal{M}_t)^2 &= \mathrm{tr}(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2 - \frac{1}{N+1}(-i\mathcal{M}_Y)^2, \quad -i\mathcal{M}_Y = \mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right), \\ \mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right) &= -N + 2\mathbf{tr}(\chi) + 2Z^2\mathbf{tr}(\chi r r^\dagger) - 4Z^2\mathbf{tr}(\chi r r^\dagger \chi) + 2Z^2 - 1, \\ \mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right)^2 &= N - 4\mathbf{tr}(\chi) + 4\mathbf{tr}(\chi\chi) + 12Z^2\mathbf{tr}(\chi r r^\dagger \chi) - 16Z^2\mathbf{tr}(\chi\chi r r^\dagger \chi) \\ &\quad -4Z^4 r^\dagger \chi\chi r \cdot \mathbf{tr}(\chi r r^\dagger) + 8Z^4 r^\dagger \chi\chi r \cdot \mathbf{tr}(\chi r r^\dagger \chi) + 1 - 4Z^4 r^\dagger \chi\chi r, \end{aligned}\right\} \tag{5.32}$$

Here we give the proof of the third identity of (5.22) as

$$
\begin{aligned}
r^{\dagger}\chi r &= \frac{1}{4Z^4}(x^{\dagger}+x^{\mathrm{T}}q^{\dagger})\chi(x+q\bar{x}) = \frac{1}{4Z^4}(x^{\dagger}\chi x+x^{\mathrm{T}}q^{\dagger}\chi q\bar{x}) \\
&= \frac{1}{4Z^4}x^{\mathrm{T}}\bar{x} = \frac{1-z^2}{4Z^4} = \frac{1-Z^2}{Z^2}.
\end{aligned}
\tag{5.33}
$$

By using the same method as the above, we can approximately calculate the quantities $r^{\dagger}\chi\chi r$ and $\mathbf{tr}(rr^{\dagger})$ as

$$
\left.
\begin{aligned}
r^{\dagger}\chi\chi r &= \frac{1}{4Z^4}(x^{\dagger}+x^{\mathrm{T}}q^{\dagger})\chi\chi(x+q\bar{x}) = \frac{1}{4Z^4}x^{\dagger}\chi x \\
&\approx \frac{1}{4Z^4}\left\{\tfrac{1}{N}\left[N+\mathbf{tr}(q^{\dagger}q)\right]\right\}^{-1}x^{\dagger}x = \frac{1-Z^2}{Z^2}<\chi>, \\
<\chi> &\stackrel{\mathrm{def}}{=} \left\{\frac{1}{N}\left[N+\mathbf{tr}(q^{\dagger}q)\right]\right\}^{-1},
\end{aligned}
\right\}
\tag{5.34}
$$

$$
\begin{aligned}
\mathbf{tr}(rr^{\dagger}) &= r^{\dagger}r = \frac{1}{4Z^4}(x^{\dagger}+x^{\mathrm{T}}q^{\dagger})(x+q\bar{x}) = \frac{1}{4Z^4}x^{\dagger}\chi^{-1}x \\
&\approx \frac{1-Z^2}{Z^2}\frac{1}{<\chi>} \stackrel{\mathrm{def}}{=} <rr^{\dagger}> .
\end{aligned}
\tag{5.35}
$$

In (5.32), approximating $\mathbf{tr}(\chi)$, $\mathbf{tr}(\chi rr^{\dagger})$, etc. by $<\chi>$, $<\chi>\mathbf{tr}(rr^{\dagger})$, etc., respectively, and using (5.34) and (5.35), $\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right)$ and $\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right)^2$ are computed as

$$
\left.
\begin{aligned}
\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right) &= 1-N+2(2Z^2-1)<\chi>, \\
\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right)^2 &= 1+N-4(2Z^2-1)<\chi>+4(2Z^4-1)<\chi>^2 .
\end{aligned}
\right\}
\tag{5.36}
$$

Substituting (5.36) into (5.31), we obtain the final form of the reduced scalar potential as

$$
\begin{aligned}
V_{\mathrm{redSC}} = &\frac{g^2_{U(1)}}{2(N+1)}\left\{\xi+1-N+2(2Z^2-1)<\chi>\right\}^2 \\
&+2\frac{g^2_{SU(N+1)}}{N+1}\left[N-2(2Z^2-1)<\chi>+\{2(N-1)Z^4+4Z^2-(N+2)\}<\chi>^2\right],
\end{aligned}
\tag{5.37}
$$

which is written in terms of the $SO(2N+1)$ parameter Z, the mean value of χ, i.e., $<\chi>$, and the Fayet-Ilipoulos parameter ξ.

In order to see the behaviour of the vacuum expectation value of the σ-fields, it is very important to analyze the form of the reduced scalar potential. From the variation of the reduced scalar potential with respect to Z and $<\chi>$, we can obtain the following relations:

$$
g^2_{U(1)}\left\{\xi+1-N+2(2Z^2-1)<\chi>\right\} - 2g^2_{SU(N+1)}\left\{1-((N-1)Z^2+1)<\chi>\right\} = 0,
\tag{5.38}
$$

$$
\begin{aligned}
&g^2_{U(1)}\left\{\xi+1-N+2(2Z^2-1)<\chi>\right\}(2Z^2-1) \\
&-2g^2_{SU(N+1)}\left[2Z^2-1-\{2(N-1)Z^4+4Z^2-(N+2)\}<\chi>\right] = 0.
\end{aligned}
\tag{5.39}
$$

Multiplying by $2Z^2-1$ for (5.38) and using (5.39), we have a g^2-independent relation

$$\begin{aligned}&\{1-((N-1)Z^2+1)<\chi>\}\,(2Z^2-1)\\&-[2Z^2-1-\{2(N-1)Z^4+4Z^2-(N+2)\}<\chi>]=0.\end{aligned} \tag{5.40}$$

This relation reads

$$(N+1)(Z^2-1)<\chi>=0, \tag{5.41}$$

from which, since $<\chi>\neq 0$, we get a very simple solution

$$Z^2=1, \quad <\chi>=\frac{1}{2}\frac{1}{g^2_{U(1)}+Ng^2_{SU(N+1)}}\left\{g^2_{U(1)}(N-1)+2g^2_{SU(N+1)}-g^2_{U(1)}\xi\right\}. \tag{5.42}$$

This solution just corresponds to the $\frac{SO(2N)}{U(N)}$ supersymmetric σ-model since $Z^2=1$. Putting this solution into (5.37), the minimization of the reduced scalar potential with respect to the Fayet-Ilipoulos parameter ξ is realized as follows:

$$\left.\begin{aligned}&V_{\mathrm{redSC}}=\frac{1}{2}\frac{N}{N+1}\frac{g^2_{U(1)}g^2_{SU(N+1)}}{g^2_{U(1)}+Ng^2_{SU(N+1)}}\left[\xi+\frac{1}{N}\{2-N(N-1)\}\right]^2+V^{\mathrm{min}}_{\mathrm{redSC}},\\&\xi_{\mathrm{min}}=N-1-2\frac{1}{N},\quad V^{\mathrm{min}}_{\mathrm{redSC}}=2\frac{g^2_{SU(N+1)}}{N+1}\left(N-\frac{1}{N}\right),\quad <\chi>_{\mathrm{min}}=\frac{1}{N}.\end{aligned}\right\} \tag{5.43}$$

To find a proper solution for the extended supersymmetric σ-model, after rescaling the Goldstone fields $\mathcal{Q}$ by a mass parameter, as van Holten et al. did [2, 13], we also introduce the $(N+1)$-dimensional matrices $\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})$, $\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})$ and $\mathcal{X}_f$ in the following forms:

$$\left.\begin{aligned}&\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})=\frac{1}{f}\delta\mathcal{B}-\delta\mathcal{A}^{\mathrm{T}}\mathcal{Q}_f-\mathcal{Q}_f\delta\mathcal{A}+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\mathcal{Q}_f,\ \mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})=-\delta\mathcal{A}^{\mathrm{T}}+f\mathcal{Q}_f\delta\mathcal{B}^\dagger,\\&\mathcal{X}_f=(1_{N+1}+f^2\mathcal{Q}_f\mathcal{Q}_f^\dagger)^{-1}=\mathcal{X}^\dagger,\ \mathcal{Q}_f=\begin{bmatrix}q & \frac{1}{f}r_f\\ -\frac{1}{f}r_f^{\mathrm{T}} & 0\end{bmatrix},\ r_f=\frac{1}{2Z^2}(x+fq\bar{x}),\ f\stackrel{\mathrm{def}}{=}\frac{1}{m_\sigma}.\end{aligned}\right\} \tag{5.44}$$

Due to the rescaling, the Killing potential $\mathcal{M}_\sigma$ is deformed as

$$\left.\begin{aligned}-i\mathcal{M}_{f\sigma}\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right)&=-\mathrm{tr}\Delta_f\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right),\\ \Delta_f\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right)&\stackrel{\mathrm{def}}{=}\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})-\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})f^2\mathcal{Q}_f^\dagger\mathcal{X}_f\\&=\left(f^2\mathcal{Q}_f\delta\mathcal{A}\mathcal{Q}_f^\dagger-\delta\mathcal{A}^{\mathrm{T}}-f\delta\mathcal{B}\mathcal{Q}_f^\dagger+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\right)\mathcal{X}_f,\end{aligned}\right\} \tag{5.45}$$

from which we obtain a f-deformed Killing potential $\mathcal{M}_{f\sigma}$

$$-i\mathcal{M}_{f\sigma\delta\mathcal{B}}=-f\mathcal{X}_f\mathcal{Q}_f,\quad -i\mathcal{M}_{f\sigma\delta\mathcal{B}^\dagger}=f\mathcal{Q}_f^\dagger\mathcal{X}_f,\quad -i\mathcal{M}_{f\sigma\delta\mathcal{A}}=1_{N+1}-2f^2\mathcal{Q}_f^\dagger\mathcal{X}_f\mathcal{Q}_f. \tag{5.46}$$

After the same algebraic manipulations, the inverse matrix $\mathcal{X}_f$ in (5.44) leads to a different form deformed from the previous one (5.16)

$$\mathcal{X}_f=\begin{bmatrix}\mathcal{Q}_{fqq^\dagger} & \mathcal{Q}_{fqr}\\ \mathcal{Q}_{fqr}^\dagger & \mathcal{Q}_{fr^\dagger r}\end{bmatrix},\quad \chi_f=(1_N+f^2qq^\dagger)^{-1}=\chi_f^\dagger, \tag{5.47}$$

where each sub-matrix is expressed by the variables q and r and the parameters f and Z as

$$\mathcal{Q}_{fqq^\dagger} = \chi_f - Z^2\chi_f(r_f r_f^\dagger - f^2 q\bar{r}_f r_f^{\mathrm{T}} q^\dagger)\chi_f, \tag{5.48}$$

$$\mathcal{Q}_{fq\bar{r}} = fZ^2\chi_f q\bar{r}_f, \quad \mathcal{Q}_{fr^\dagger r} = Z^2, \tag{5.49}$$

which are derived in Appendix E. Substituting (5.44) and (5.47) into (5.46) and introducing a f-deformed auxiliary function $\lambda_f = r_f r_f^\dagger - f^2 q\bar{r}_f r_f^{\mathrm{T}} q^\dagger = \lambda_f^\dagger$, we can get the f-deformed Killing potential $\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as,

$$-i\mathcal{M}_{f\sigma\delta\mathcal{A}} =$$

$$\begin{bmatrix} \begin{array}{c} 1_N - 2q^\dagger\chi_f q + 2Z^2\left(q^\dagger\chi_f\lambda_f\chi_f q + q^\dagger\chi_f q\bar{r}_f r_f^{\mathrm{T}}\right. \\ \left. + \bar{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f q - \frac{1}{f^2}\bar{r}_f r_f^{\mathrm{T}}\right) \end{array} & \begin{array}{c} -2\frac{1}{f}q^\dagger\chi_f r_f + 2\frac{1}{f}Z^2\left(q^\dagger\chi_f\lambda_f\chi_f r_f\right. \\ \left. + \bar{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f r_f\right) \end{array} \\ \begin{array}{c} -2\frac{1}{f}r_f^\dagger\chi_f q + 2\frac{1}{f}Z^2\left(r_f^\dagger\chi_f\lambda_f\chi_f q\right. \\ \left. + r_f^\dagger\chi_f q\bar{r}_f r_f^{\mathrm{T}}\right) \end{array} & 1 - 2\frac{1}{f^2}r_f^\dagger\chi_f r_f + 2\frac{1}{f^2}Z^2 r_f^\dagger\chi_f\lambda_f\chi_f r_f \end{bmatrix}, \tag{5.50}$$

in which each block-matrix satisfy the following identities and relations:

$$r_f^{\mathrm{T}} q^\dagger\chi_f r_f = 0, \quad r_f^\dagger\chi_f q\bar{r}_f = 0, \quad r_f^\dagger\chi_f r_f = \frac{1-Z^2}{Z^2}, \quad r_f^\dagger\chi_f\lambda_f\chi_f r_f = \left(\frac{1-Z^2}{Z^2}\right)^2, \tag{5.51}$$

$$1 - 2\frac{1}{f^2}r_f^\dagger\chi_f r_f + 2\frac{1}{f^2}Z^2 r_f^\dagger\chi_f\lambda_f\chi_f r_f = \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2}, \tag{5.52}$$

$$\chi_f\lambda_f\chi_f r_f = \frac{1-Z^2}{Z^2}\chi_f r_f, \quad r_f^\dagger\chi_f\lambda_f\chi_f = \frac{1-Z^2}{Z^2}r_f^\dagger\chi_f, \quad q^\dagger\chi_f q = \frac{1}{f^2}(1_N - \bar{\chi}_f). \tag{5.53}$$

Using these identities and relations, we get a more compact form of the f-deformed Killing potential $\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as,

$$-i\mathcal{M}_{f\sigma\delta\mathcal{A}} = \begin{bmatrix} 1_N - 2q^\dagger\chi_f q + 2Z^2\left(q^\dagger\chi_f r_f r_f^\dagger\chi_f q - \frac{1}{f^2}\bar{\chi}_f\bar{r}_f r_f^{\mathrm{T}}\bar{\chi}_f\right) & -2\frac{1}{f}Z^2 q^\dagger\chi_f r_f \\ -2\frac{1}{f}Z^2 r_f^\dagger\chi_f q & \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2} \end{bmatrix}. \tag{5.54}$$

Owing to the rescaling, the f-deformed reduced scalar potential is written as follows:

$$\left.\begin{aligned} V_{f\mathrm{redSC}} &= \frac{g_{U(1)}^2}{2(N+1)}\left(\xi - i\mathcal{M}_{fY}\right)^2 + \frac{g_{SU(N+1)}^2}{2}\mathrm{tr}\left(-i\mathcal{M}_{ft}\right)^2, \\ \mathrm{tr}(-i\mathcal{M}_{ft})^2 &= \mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2 - \frac{1}{N+1}(-i\mathcal{M}_{fY})^2, \quad -i\mathcal{M}_{fY} = \mathrm{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right), \end{aligned}\right\} \tag{5.55}$$

in which each f-deformed Killing potential is calculated straight forwardly in the following forms:

$$\begin{aligned} \mathrm{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right) = &\left(1 - 2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}\mathbf{tr}(\chi_f) + 2\frac{1}{f^2}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger) - 4\frac{1}{f^2}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f) \\ &+ \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2}, \end{aligned} \tag{5.56}$$

$$
\begin{aligned}
\operatorname{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right)^2 &= N - 4\frac{1}{f^2}(1-\frac{1}{f^2})N - 4\frac{1}{f^4}\mathbf{tr}(\chi_f) + 4\frac{1}{f^4}\mathbf{tr}(\chi_f\chi_f) \\
&+4\frac{1}{f^2}(1-\frac{1}{f^2})Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger) - 4\frac{1}{f^2}(1-\frac{1}{f^2})Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger \chi_f) \\
&+12\frac{1}{f^4}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger \chi_f) - 16\frac{1}{f^4}Z^2\mathbf{tr}(\chi_f\chi_f r_f r_f^\dagger \chi_f) \\
&-4\frac{1}{f^4}Z^4 r_f^\dagger \chi_f\chi_f r_f \cdot \mathbf{tr}(\chi_f r_f r_f^\dagger) + 8\frac{1}{f^4}Z^4 r_f^\dagger \chi_f\chi_f r_f \cdot \mathbf{tr}(\chi_f r_f r_f^\dagger \chi_f) \\
&+\frac{1}{f^4} + 2\frac{1}{f^2}(1-\frac{1}{f^2})(2Z^2-1) + (1-\frac{1}{f^2})^2 - 4\frac{1}{f^4}Z^4 r_f^\dagger \chi_f\chi_f r_f.
\end{aligned}
\tag{5.57}
$$

The identity below is also derived

$$
\begin{aligned}
r_f^\dagger \chi_f r_f &= \frac{1}{4Z^4}(x^\dagger + f x^{\mathrm{T}} q^\dagger)\chi_f(x + f q\bar{x}) = \frac{1}{4Z^4}(x^\dagger\chi_f x + x^{\mathrm{T}} q^\dagger \chi_f q\bar{x}) \\
&= \frac{1}{4Z^4}x^{\mathrm{T}}\bar{x} = \frac{1-z^2}{4Z^4} = \frac{1-Z^2}{Z^2},
\end{aligned}
\tag{5.58}
$$

and approximate formulas for the quantities $r_f^\dagger \chi_f \chi_f r_f$ and $\mathbf{tr}(r_f r_f^\dagger)$ can be calculated as

$$
\left.
\begin{aligned}
r_f^\dagger \chi_f\chi_f r_f &= \frac{1}{4Z^4}(x^\dagger + f x^{\mathrm{T}} q^\dagger)\chi_f\chi_f(x + f q\bar{x}) = \frac{1}{4Z^4}x^\dagger\chi_f x \\
&\approx \frac{1}{4Z^4}\left\{\tfrac{1}{N}\left[N + f^2\mathbf{tr}(q^\dagger q)\right]\right\}^{-1} x^\dagger x = \frac{1-Z^2}{Z^2} < \chi_f >, \\
< \chi_f > &\stackrel{\mathrm{def}}{=} \left\{\frac{1}{N}\left[N + f^2\mathbf{tr}(q^\dagger q)\right]\right\}^{-1},
\end{aligned}
\right\}
\tag{5.59}
$$

$$
\begin{aligned}
\mathbf{tr}(r_f r_f^\dagger) = r_f^\dagger r_f &= \frac{1}{4Z^4}(x^\dagger + f x^{\mathrm{T}} q^\dagger)(x + f q\bar{x}) = \frac{1}{4Z^4}x^\dagger \chi_f^{-1} x \\
&\approx \frac{1-Z^2}{Z^2}\frac{1}{< \chi_f >} \stackrel{\mathrm{def}}{=} < r_f r_f^\dagger > .
\end{aligned}
\tag{5.60}
$$

In (5.56) and (5.57), approximating $\mathbf{tr}(\chi_f)$, $\mathbf{tr}(\chi_f r_f r_f^\dagger)$, etc. by $<\chi_f>$, $<\chi_f>\mathbf{tr}(r_f r_f^\dagger)$, etc., respectively, and using (5.59) and (5.60), $\operatorname{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right)$ and $\operatorname{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right)^2$ are computed as

$$
\left.
\begin{aligned}
\operatorname{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right) &= 1 + \left(1 - 2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}(2Z^2-1) < \chi_f >, \\
\operatorname{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right)^2 &= 1 + N - 4\frac{1}{f^2}\left(1 - \frac{1}{f^2}\right)N \\
&\quad - 4\frac{1}{f^2}\left\{\frac{1}{f^2}(2Z^2-1) - \left(1 - \frac{1}{f^2}\right)Z^2\right\} < \chi_f > + 4\frac{1}{f^4}(2Z^4-1) < \chi_f >^2 .
\end{aligned}
\right\}
\tag{5.61}
$$

Substituting (5.61) into (5.55), we obtain the f-deformed reduced scalar potential as

$$
\begin{aligned}
V_{f\mathrm{redSC}} &= \frac{g_{U(1)}^2}{2(N+1)} \left[\xi + 1 + \left(1 - 2\frac{1}{f^2}\right) N + 2\frac{1}{f^2}(2Z^2 - 1) <\chi_f> \right]^2 \\
&+2\frac{g_{SU(N+1)}^2}{N+1}\frac{1}{f^2}\left[\frac{1}{f^2}N - \left\{\left(1 - \frac{1}{f^2}\right)N + \left(1 + 3\frac{1}{f^2}\right)\right\} Z^2 <\chi_f> \right. \\
&\left. + \left\{\left(1 - \frac{1}{f^2}\right)N + \left(1 + \frac{1}{f^2}\right)\right\} <\chi_f> + \frac{1}{f^2}\left\{2(N-1)Z^4 + 4Z^2 - (N+2)\right\} <\chi_f>^2 \right],
\end{aligned}
\tag{5.62}
$$

and, from the variation of this with respect to Z and $<\chi_f>$, we get the following relations:

$$
\begin{aligned}
& g_{U(1)}^2 \left\{ \xi + 1 + \left(1 - 2\frac{1}{f^2}\right) N + 2\frac{1}{f^2}(2Z^2 - 1) <\chi_f> \right\} \\
& -2g_{SU(N+1)}^2 \left[\frac{1}{4}\left\{\left(1 - \frac{1}{f^2}\right)N + \left(1 + 3\frac{1}{f^2}\right)\right\} - \frac{1}{f^2}\left\{(N-1)Z^2 + 1\right\} <\chi_f> \right] = 0,
\end{aligned}
\tag{5.63}
$$

$$
\begin{aligned}
& g_{U(1)}^2 \left[\xi + 1 + \left(1 - 2\frac{1}{f^2}\right) N + 2\frac{1}{f^2}(2Z^2 - 1) <\chi_f> \right] (2Z^2 - 1) \\
& -2g_{SU(N+1)}^2 \left[\frac{1}{2}\left\{\left(1 - \frac{1}{f^2}\right)N + \left(1 + 3\frac{1}{f^2}\right)\right\} Z^2 - \frac{1}{2}\left(1 - \frac{1}{f^2}\right)N - \frac{1}{2}\left(1 + \frac{1}{f^2}\right) \right. \\
& \qquad \left. - \frac{1}{f^2}\left\{2(N-1)Z^4 + 4Z^2 - (N+2)\right\} <\chi_f> \right] = 0.
\end{aligned}
\tag{5.64}
$$

Multiplying by $(2Z^2-1)$ for (5.63) and using (5.64), we have a g^2-independent relation

$$
\begin{aligned}
& \left[\frac{1}{4}\left\{\left(1 - \frac{1}{f^2}\right)N + 1 + 3\frac{1}{f^2}\right\} - \frac{1}{f^2}\left\{(N-1)Z^2 + 1\right\} <\chi_f> \right] (2Z^2-1) \\
& - \left[\frac{1}{2}\left\{\left(1 - \frac{1}{f^2}\right)N + \left(1 + 3\frac{1}{f^2}\right)\right\} Z^2 - \frac{1}{2}\left(1 - \frac{1}{f^2}\right)N - \frac{1}{2}\left(1 + \frac{1}{f^2}\right) \right. \\
& \qquad \left. - \frac{1}{f^2}\left\{2(N-1)Z^4 + 4Z^2 - (N+2)\right\} <\chi_f> \right] = 0.
\end{aligned}
\tag{5.65}
$$

This relation reads

$$
(N+1)\left\{4(Z^2-1) <\chi_f> - \left(1 - f^2\right)\right\} = 0,
\tag{5.66}
$$

through which due to $0 \le Z^2 \le 1$ and $<\chi_f>> 0$, the rescaling parameter f is shown to satisfy $f^2 \ge 1$. From (5.66), since $<\chi_f> \neq 0$, we can finally reach our ultimate goal of proper solutions for Z^2 and $<\chi_f>$ as

$$
\left.
\begin{aligned}
& Z^2 = 1 + \frac{1}{2}(1 - f^2) \frac{g_{U(1)}^2 + N g_{SU(N+1)}^2}{g_{U(1)}^2 \left\{(2-f^2)N - 1\right\} - g_{SU(N+1)}^2 \left\{(1-f^2)N - 2\right\} - g_{U(1)}^2 f^2 \xi}, \\
& <\chi_f> = \frac{1}{2} \frac{1}{g_{U(1)}^2 + N g_{SU(N+1)}^2} \left[g_{U(1)}^2 \left\{(2-f^2)N - 1\right\} - g_{SU(N+1)}^2 \left\{(1-f^2)N - 2\right\} - g_{U(1)}^2 f^2 \xi \right].
\end{aligned}
\right\}
\tag{5.67}
$$

This is just the solution for the $\frac{SO(2N+2)}{U(N+1)}$ supersymmetric σ-model. The solution (5.67), if $f^2 = 1$, reduces to a simple solution (5.42).

6 Discussions and concluding remarks

In order to find a proper solution for the extended $\frac{SO(2N+2)}{U(N+1)}$ supersymmetric σ-model, the minimization of the f-deformed reduced scalar potential has been made after rescaling Goldstone fields by a mass parameter. Then the proper solutions for Z^2 and $<\chi_f>$ (5.67) have been produced. In the course of producing such solutions, the Fayet-Ilipoulos term has made a crucial role. Substituting (5.67) into (5.62), the minimization of the f-deformed reduced scalar potential with respect to the Fayet-Ilipoulos parameter ξ is realized as follows:

$$\left.\begin{array}{l} V_{f\mathrm{redSC}} = \dfrac{1}{2}\dfrac{N}{N+1}\dfrac{g^2_{U(1)}g^2_{SU(N+1)}}{g^2_{U(1)}+Ng^2_{SU(N+1)}}\left[\xi+\dfrac{1}{N}\left\{\left(1-2\dfrac{1}{f^2}\right)N^2+N+2\dfrac{1}{f^2}\right\}\right]^2+V^{\min}_{f\mathrm{redSC}}, \\ \xi_{\min} = -\left(1-2\dfrac{1}{f^2}\right)N-1-2\dfrac{1}{f^2}\dfrac{1}{N}, \\ V^{\min}_{f\mathrm{redSC}} = 2\dfrac{g^2_{SU(N+1)}}{N+1}\left[\left\{\dfrac{1}{f^4}+\dfrac{1}{8}\left(1-\dfrac{1}{f^2}\right)^2\right\}N+\dfrac{1}{8}\left(1-\dfrac{1}{f^2}\right)^2-\dfrac{1}{f^4}\dfrac{1}{N}\right]. \end{array}\right\} \quad (6.1)$$

Thus we get the minimized f-deformed reduced scalar potential $V^{\min}_{f\mathrm{redSC}}$ if we choose the Fayet-Ilipoulos parameter ξ to be $\xi_{\min}$. Putting this $\xi_{\min}$ into (5.67), we have

$$\left.\begin{array}{l} Z^2_{\min} = \dfrac{1}{2}+\dfrac{1}{2}\dfrac{1}{N}\dfrac{1}{\frac{1}{2}(f^2-1)+\frac{1}{N}}, \\ <\chi_f>_{\min} = \dfrac{1}{2}(f^2-1)+\dfrac{1}{N}, \quad f^2 \geq 1. \end{array}\right\} \quad (6.2)$$

Equations (6.1) and (6.2), if $f^2=1$, reduce to (5.43).

In this paper, we have given an extended supersymmetric σ-model on the Kähler coset space $\frac{G}{H}=\frac{SO(2N+2)}{U(N+1)}$, basing on the $SO(2N+1)$ Lie algebra of the fermion operators. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables [17], we have investigated a new aspect of the extended supersymmetric σ-model which has never been seen in the usual supersymmetric σ-model on the Kähler coset space $\frac{SO(2N)}{U(N)}$ given by van Holten et al. [2]. We have constructed a Killing potential, the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. To our great surprise, the Killing potential is equivalent with the generalized density matrix which is a useful tool to study fermion many-body problems. Its diagonal-block part is related to a reduced scalar potential with the Fayet-Ilipoulos term. The reduced and the f-deformed reduced scalar potentials have been optimized in order to see the behaviour of the vacuum expectation value of the σ-model fields. We have got, if $f^2=1$, a simple solution $Z^2=1$ corresponding to the $\frac{SO(2N)}{U(N)}$ supersymmetric σ-model and the proper solutions for Z^2 and $<\chi_f>$. The Fayet-Ilipoulos term has made an important role to get such solutions.

Finally, we have given bosonization of the $SO(2N+2)$ Lie operators, vacuum functions and differential forms for their bosons expressed in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, a $U(1)$ phase and the corresponding Kähler potential. This provides a powerful tool for describing the Goldstone bosons but accompanying fermionic modes in the present model. The effectiveness of $\frac{SO(2N+2)}{U(N+1)}$ Kähler manifold is expected to open a new field for exploration of low-energy elementary particle physics by the supersymmetric σ-model.

Acknowledgements

One of the authors (S. N.) would like to express his sincere thanks to Professor Alex H. Blin for kind and warm hospitality extended to him at the Centro de Física Teórica, Universidade de Coimbra, Portugal. This work was supported by the Portuguese Project POCTI/FIS/451/94. The authors thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-07-05 on "String Theory and Quantum Field Theory" were useful to complete this work.

Appendix

A Bosonization of SO(2N+2) Lie operators

Consider a fermion state vector $|\,\Psi>$ corresponding to a function $\Psi(\mathcal{G})$ in $\mathcal{G} \in SO(2N+2)$:

$$|\,\Psi> = \int U(\mathcal{G})|\,0><0\,|U^\dagger(\mathcal{G})|\,\Psi>d\mathcal{G} = \int U(\mathcal{G})|\,0>\Psi(\mathcal{G})d\mathcal{G}. \tag{A.1}$$

The $\mathcal{G}$ is given by (3.11) and (3.12) and the $d\mathcal{G}$ is an invariant group integration. When an infinitesimal operator $\mathbb{I}_{\mathcal{G}}+\delta\widehat{\mathcal{G}}$ and a corresponding infinitesimal unitary operator $U(1_{2N+2}+\delta\mathcal{G})$ is operated on $|\,\Psi>$, using $U^{-1}(1_{2N+2}+\delta\mathcal{G})=U(1_{2N+2}-\delta\mathcal{G})$, it transforms $|\,\Psi>$ as

$$\begin{aligned}
&U(1_{2N+2}-\delta\mathcal{G})|\,\Psi> = (\mathbb{I}_{\mathcal{G}}-\delta\widehat{\mathcal{G}})|\,\Psi> = \int U(\mathcal{G})|\,0><0\,|U^\dagger((1_{2N+2}+\delta\mathcal{G})\mathcal{G})|\,\Psi>d\mathcal{G} \\
&= \int U(\mathcal{G})|\,0>\Psi((1_{2N+2}+\delta\mathcal{G})\mathcal{G})d\mathcal{G} = \int U(\mathcal{G})|\,0>(1_{2N+2}+\delta\boldsymbol{\mathcal{G}})\Psi(\mathcal{G})d\mathcal{G},
\end{aligned} \tag{A.2}$$

$$\left.\begin{aligned}
&1_{2N+2}+\delta\mathcal{G} = \begin{bmatrix} 1_{N+1}+\delta\mathcal{A} & \delta\bar{\mathcal{B}} \\ \delta\mathcal{B} & 1_{N+1}+\delta\bar{\mathcal{A}} \end{bmatrix}, \ \delta\mathcal{A}^\dagger = -\delta\mathcal{A}, \ \mathrm{tr}\delta\mathcal{A}=0, \ \delta\mathcal{B} = -\delta\mathcal{B}^{\mathrm{T}}, \\
&\delta\widehat{\mathcal{G}} = \delta\mathcal{A}^p{}_q E^q{}_p + \frac{1}{2}\left(\delta\mathcal{B}_{pq}E^{qp} + \delta\bar{\mathcal{B}}_{pq}E_{qp}\right), \ \delta\boldsymbol{\mathcal{G}} = \delta\mathcal{A}^p{}_q \boldsymbol{\mathcal{E}^q{}_p} + \frac{1}{2}\left(\delta\mathcal{B}_{pq}\boldsymbol{\mathcal{E}^{qp}} + \delta\bar{\mathcal{B}}_{pq}\boldsymbol{\mathcal{E}_{qp}}\right).
\end{aligned}\right\} \tag{A.3}$$

Equation (A.2) shows that the operation of $\mathbb{I}_{\mathcal{G}}-\delta\widehat{\mathcal{G}}$ on the $|\,\Psi>$ in the fermion space corresponds to the left multiplication by $1_{2N+2}+\delta\mathcal{G}$ for the variable of the $\mathcal{G}$ of the function $\Psi(\mathcal{G})$. For a small parameter ϵ, we obtain a representation on the $\Psi(\mathcal{G})$ as

$$\rho(e^{\epsilon\delta\mathcal{G}})\Psi(\mathcal{G}) = \Psi(e^{\epsilon\delta\mathcal{G}}\mathcal{G}) = \Psi(\mathcal{G}+\epsilon\delta\mathcal{G}\mathcal{G}) = \Psi(\mathcal{G}+d\mathcal{G}), \tag{A.4}$$

which leads us to a relation $d\mathcal{G} = \epsilon\delta\mathcal{G}\mathcal{G}$. From this, we express it explicitly as,

$$\left.\begin{aligned}
&d\mathcal{G} = \begin{bmatrix} d\mathcal{A} & d\bar{\mathcal{B}} \\ d\mathcal{B} & d\bar{\mathcal{A}} \end{bmatrix} = \epsilon\begin{bmatrix} \delta\mathcal{A}\mathcal{A}+\delta\bar{\mathcal{B}}\mathcal{B} & \delta\mathcal{A}\bar{\mathcal{B}}+\delta\bar{\mathcal{B}}\bar{\mathcal{A}} \\ \delta\mathcal{B}\mathcal{A}+\delta\bar{\mathcal{A}}\mathcal{B} & \delta\bar{\mathcal{A}}\bar{\mathcal{A}}+\delta\mathcal{B}\bar{\mathcal{B}} \end{bmatrix}, \\
&d\mathcal{A} = \epsilon\frac{\partial\mathcal{A}}{\partial\epsilon} = \epsilon(\delta\mathcal{A}\mathcal{A}+\delta\bar{\mathcal{B}}\mathcal{B}), \quad d\mathcal{B} = \epsilon\frac{\partial\mathcal{B}}{\partial\epsilon} = \epsilon(\delta\mathcal{B}\mathcal{A}+\delta\bar{\mathcal{A}}\mathcal{B}).
\end{aligned}\right\} \tag{A.5}$$

A differential representation of $\rho(\delta\mathcal{G})$, $d\rho(\delta\mathcal{G})$, is given as

$$d\rho(\delta\mathcal{G})\Psi(\mathcal{G}) = \left[\frac{\partial\mathcal{A}^p{}_q}{\partial\epsilon}\frac{\partial}{\partial\mathcal{A}^p{}_q} + \frac{\partial\mathcal{B}_{pq}}{\partial\epsilon}\frac{\partial}{\partial\mathcal{B}_{pq}} + \frac{\partial\bar{\mathcal{A}}^p{}_q}{\partial\epsilon}\frac{\partial}{\partial\bar{\mathcal{A}}^p{}_q} + \frac{\partial\bar{\mathcal{B}}_{pq}}{\partial\epsilon}\frac{\partial}{\partial\bar{\mathcal{B}}_{pq}}\right]\Psi(\mathcal{G}). \tag{A.6}$$

Substituting (A.5) into (A.6), we can get explicit forms of the differential representation

$$d\rho(\delta\mathcal{G})\Psi(\mathcal{G}) = \delta\boldsymbol{\mathcal{G}}\Psi(\mathcal{G}), \tag{A.7}$$

where each operator in $\delta\boldsymbol{\mathcal{G}}$ is expressed in a differential form as

$$\left.\begin{aligned}
&\boldsymbol{\mathcal{E}^p{}_q} = \bar{\mathcal{B}}_{pr}\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{B}_{pr}} - \bar{\mathcal{A}}^q{}_r\frac{\partial}{\partial\bar{\mathcal{A}}^p{}_r} + \mathcal{A}^p{}_r\frac{\partial}{\partial\mathcal{A}^q{}_r} = \boldsymbol{\mathcal{E}^{q\dagger}_p}, \\
&\boldsymbol{\mathcal{E}_{pq}} = \bar{\mathcal{A}}^p{}_r\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{A}^p{}_r} - \bar{\mathcal{A}}^q{}_r\frac{\partial}{\partial\bar{\mathcal{B}}_{pr}} + \mathcal{B}_{pr}\frac{\partial}{\partial\mathcal{A}^q{}_r} = \boldsymbol{\mathcal{E}^{qp\dagger}}, \\
&\boldsymbol{\mathcal{E}^{p\dagger}_q} = -\bar{\boldsymbol{\mathcal{E}}}\boldsymbol{{}^p{}_q}, \qquad \boldsymbol{\mathcal{E}^\dagger_{pq}} = -\bar{\boldsymbol{\mathcal{E}}}_{\boldsymbol{pq}}, \qquad \boldsymbol{\mathcal{E}_{pq}} = -\boldsymbol{\mathcal{E}_{qp}}.
\end{aligned}\right\} \tag{A.8}$$

We define the boson operators $\boldsymbol{\mathcal{A}^p}_{\boldsymbol{q}}$, $\bar{\boldsymbol{\mathcal{A}}}^{\boldsymbol{p}}{}_{\boldsymbol{q}}$, etc., from every variable $\mathcal{A}^p{}_q$, $\bar{\mathcal{A}}^p{}_q$, etc., as

$$\left.\begin{array}{l}\boldsymbol{\mathcal{A}} \stackrel{\mathrm{def}}{=} \dfrac{1}{\sqrt{2}}\left(\mathcal{A}+\dfrac{\partial}{\partial\bar{\mathcal{A}}}\right),\quad \boldsymbol{\mathcal{A}}^{\dagger} \stackrel{\mathrm{def}}{=} \dfrac{1}{\sqrt{2}}\left(\bar{\mathcal{A}}-\dfrac{\partial}{\partial\mathcal{A}}\right),\\[2ex] \bar{\boldsymbol{\mathcal{A}}} \stackrel{\mathrm{def}}{=} \dfrac{1}{\sqrt{2}}\left(\bar{\mathcal{A}}+\dfrac{\partial}{\partial\mathcal{A}}\right),\quad \boldsymbol{\mathcal{A}}^{\mathrm{T}} \stackrel{\mathrm{def}}{=} \dfrac{1}{\sqrt{2}}\left(\mathcal{A}-\dfrac{\partial}{\partial\bar{\mathcal{A}}}\right),\\[2ex] [\boldsymbol{\mathcal{A}},\ \boldsymbol{\mathcal{A}}^{\dagger}]=1,\quad [\bar{\boldsymbol{\mathcal{A}}},\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}]=1,\\[2ex] [\boldsymbol{\mathcal{A}},\ \bar{\boldsymbol{\mathcal{A}}}]=[\boldsymbol{\mathcal{A}},\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}]=0,\quad [\boldsymbol{\mathcal{A}}^{\dagger},\ \bar{\boldsymbol{\mathcal{A}}}]=[\boldsymbol{\mathcal{A}}^{\dagger},\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}]=0,\end{array}\right\} \tag{A.9}$$

where $\mathcal{A}$ is a complex variable. Similar definitions hold for $\mathcal{B}$ in order to define the boson operators $\boldsymbol{\mathcal{B}_{pq}}$, $\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{pq}}$, etc. By noting the relations

$$\bar{\mathcal{A}}\frac{\partial}{\partial\bar{\mathcal{A}}}-\mathcal{A}\frac{\partial}{\partial\mathcal{A}}=\boldsymbol{\mathcal{A}}^{\dagger}\boldsymbol{\mathcal{A}}-\boldsymbol{\mathcal{A}}^{\mathrm{T}}\bar{\boldsymbol{\mathcal{A}}},\qquad \bar{\mathcal{A}}\frac{\partial}{\partial\bar{\mathcal{B}}}-\mathcal{B}\frac{\partial}{\partial\mathcal{A}}=\boldsymbol{\mathcal{A}}^{\dagger}\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{B}}^{\mathrm{T}}\bar{\boldsymbol{\mathcal{A}}}, \tag{A.10}$$

the differential operators (A.8) can be converted into a boson operator representation

$$\left.\begin{array}{l}\boldsymbol{\mathcal{E}^p}_{\boldsymbol{q}}=\boldsymbol{\mathcal{B}}^{\dagger}_{\boldsymbol{pr}}\boldsymbol{\mathcal{B}_{qr}}-\boldsymbol{\mathcal{B}}^{\mathrm{T}}_{\boldsymbol{qr}}\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{pr}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{q}\dagger}{}_{\boldsymbol{r}}\boldsymbol{\mathcal{A}^p}_{\boldsymbol{r}}+\boldsymbol{\mathcal{A}}^{\boldsymbol{p}\mathrm{T}}{}_{\boldsymbol{r}}\bar{\boldsymbol{\mathcal{A}}}^{\boldsymbol{q}}{}_{\boldsymbol{r}}=\boldsymbol{\mathcal{B}}^{\dagger}_{\boldsymbol{p}\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{q}\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{q}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{A}^p}_{\tilde{\boldsymbol{r}}},\\[2ex] \boldsymbol{\mathcal{E}_{pq}}=\boldsymbol{\mathcal{A}}^{\boldsymbol{p}\dagger}{}_{\boldsymbol{r}}\boldsymbol{\mathcal{B}_{qr}}-\boldsymbol{\mathcal{B}}^{\mathrm{T}}_{\boldsymbol{qr}}\bar{\boldsymbol{\mathcal{A}}}^{\boldsymbol{p}}{}_{\boldsymbol{r}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{q}\dagger}{}_{\boldsymbol{r}}\boldsymbol{\mathcal{B}_{pr}}+\boldsymbol{\mathcal{B}}^{\mathrm{T}}_{\boldsymbol{pr}}\bar{\boldsymbol{\mathcal{A}}}^{\boldsymbol{q}}{}_{\boldsymbol{r}}=\boldsymbol{\mathcal{A}}^{\boldsymbol{p}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{q}\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{q}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{p}\tilde{\boldsymbol{r}}},\end{array}\right\} \tag{A.11}$$

by using the notation $\boldsymbol{\mathcal{A}}^{\boldsymbol{p}\mathrm{T}}{}_{\boldsymbol{r+N}}=\boldsymbol{\mathcal{B}}^{\dagger}_{\boldsymbol{pr}}$ and $\boldsymbol{\mathcal{B}}^{\mathrm{T}}_{\boldsymbol{pr+N}}=\boldsymbol{\mathcal{A}}^{\boldsymbol{p}\dagger}{}_{\boldsymbol{r}}$ to use a suffix $\tilde{r}$ running from 0 to N and from N to $2N$. Then we have the boson images of the fermion $SO(2N+1)$ Lie operators as

$$\left.\begin{array}{l}\boldsymbol{E}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}=\boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}=\boldsymbol{\mathcal{B}}^{\dagger}_{\boldsymbol{\alpha}\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{\beta}\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{\beta}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{A}}^{\boldsymbol{\alpha}}{}_{\tilde{\boldsymbol{r}}},\\[2ex] \boldsymbol{E}_{\boldsymbol{\alpha\beta}}=\boldsymbol{\mathcal{E}}_{\boldsymbol{\alpha\beta}}=\boldsymbol{\mathcal{A}}^{\boldsymbol{\alpha}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{\beta}\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{A}}^{\boldsymbol{\beta}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}\tilde{\boldsymbol{r}}},\\[2ex] \boldsymbol{c}_{\boldsymbol{\alpha}}=\boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha 0}}-\boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{0}}=\boldsymbol{\mathcal{A}}^{\boldsymbol{\alpha}\dagger}{}_{\tilde{\boldsymbol{r}}}(\boldsymbol{\mathcal{A}}^{\boldsymbol{0}}{}_{\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{B}}_{\boldsymbol{0}\tilde{\boldsymbol{r}}})+(\boldsymbol{\mathcal{A}}^{\boldsymbol{0}\dagger}{}_{\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{B}}^{\dagger}_{\boldsymbol{0}\tilde{\boldsymbol{r}}})\boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}\tilde{\boldsymbol{r}}}\\ \qquad=\sqrt{2}\left(\boldsymbol{\mathcal{A}}^{\boldsymbol{\alpha}\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{Y}}_{\tilde{\boldsymbol{r}}}+\boldsymbol{\mathcal{Y}}^{\dagger}{}_{\tilde{\boldsymbol{r}}}\boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}\tilde{\boldsymbol{r}}}\right),\quad \boldsymbol{\mathcal{Y}}_{\tilde{\boldsymbol{r}}}\stackrel{\mathrm{def}}{=}\dfrac{1}{\sqrt{2}}(\boldsymbol{\mathcal{A}}^{\boldsymbol{0}}{}_{\tilde{\boldsymbol{r}}}-\boldsymbol{\mathcal{B}}_{\boldsymbol{0}\tilde{\boldsymbol{r}}}).\end{array}\right\} \tag{A.12}$$

and $\boldsymbol{\mathcal{E}}^{\boldsymbol{0}}{}_{\boldsymbol{0}}=0$ and $\boldsymbol{\mathcal{E}_{00}}=0$. The last representation for $\boldsymbol{c}_{\boldsymbol{\alpha}}$ in (A.12) involves, in addition to the original $\boldsymbol{A}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$ and $\boldsymbol{B}_{\boldsymbol{\alpha\beta}}$ bosons and $\boldsymbol{X}_{\boldsymbol{\alpha}}$ bosons, their complex conjugate bosons and the $\boldsymbol{\mathcal{Y}}_{\tilde{\boldsymbol{r}}}$ bosons. The complex conjugate bosons arise from the use of matrix $\mathcal{G}$ as the variables of representation and the $\boldsymbol{\mathcal{Y}}_{\tilde{\boldsymbol{r}}}$ bosons arise from extension of algebra from $SO(2N)$ to $SO(2N+1)$ and embedding of the $SO(2N+1)$ into $SO(2N+2)$.

Using the relations

$$\frac{\partial}{\partial\mathcal{A}^p{}_q}\det\mathcal{A}=(\mathcal{A}^{-1})_p{}^q\det\mathcal{A},\qquad \frac{\partial}{\partial\mathcal{A}^p{}_q}(\mathcal{A}^{-1})_s{}^r=-(\mathcal{A}^{-1})_s{}^q(\mathcal{A}^{-1})_p{}^r, \tag{A.13}$$

we get the relations which are valid when operated on functions on the right coset $\frac{SO(2N+2)}{SU(N+1)}$

$$\left.\begin{array}{l}\dfrac{\partial}{\partial\mathcal{B}_{pq}}=\sum_{r<p}(\mathcal{A}^{-1})_r{}^q\dfrac{\partial}{\partial\mathcal{Q}_{pr}},\\ \dfrac{\partial}{\partial\mathcal{A}^p{}_q}=-\sum_{s<r<p}\mathcal{Q}_{rp}(\mathcal{A}^{-1})_s{}^q\dfrac{\partial}{\partial\mathcal{Q}_{rs}}-\dfrac{i}{2}(\mathcal{A}^{-1})_p{}^q\dfrac{\partial}{\partial\tau},\end{array}\right\} \tag{A.14}$$

from which we can derive the expressions (C.1).

B Vacuum function for bosons

We show here that the function $\Phi_{00}(\mathcal{G})$ in $\mathcal{G}\in SO(2N+2)$ corresponds to the free fermion vacuum function in the physical fermion space. Then the $\Phi_{00}(\mathcal{G})$ must satisfy the conditions

$$\left(\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}{}_{\boldsymbol{q}}+\frac{1}{2}\delta_{pq}\right)\Phi_{00}(\mathcal{G})=\boldsymbol{\mathcal{E}}_{\boldsymbol{pq}}\Phi_{00}(\mathcal{G})=0,\quad \Phi_{00}(1_{2N+2})=1. \tag{B.1}$$

The vacuum function $\Phi_{00}(\mathcal{G})$ which satisfy (B.1) is given by $\Phi_{00}(\mathcal{G})=[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}$, the proof of which is made easily as follows:

$$\begin{aligned}
&\left(\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}{}_{\boldsymbol{q}}+\frac{1}{2}\delta_{pq}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}\\
&=\frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}+\left(\bar{\mathcal{B}}_{pr}\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}}-\mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{B}_{pr}}-\bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}}+\mathcal{A}^{p}{}_{r}\frac{\partial}{\partial\mathcal{A}^{q}{}_{r}}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}\\
&=\frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}-\bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}=\frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}-\frac{1}{2}\frac{1}{[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}}\bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}}\det(\bar{\mathcal{A}})\\
&=\frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}-\frac{1}{2}\frac{1}{[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}}(\bar{\mathcal{A}}\bar{\mathcal{A}}^{-1})_{qp}\det(\bar{\mathcal{A}})=0,
\end{aligned}\tag{B.2}$$

$$\boldsymbol{\mathcal{E}}_{\boldsymbol{pq}}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}=\left(\bar{\mathcal{A}}^{p}{}_{r}\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}}-\mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{A}^{p}{}_{r}}-\bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{B}}_{pr}}+\mathcal{B}_{pr}\frac{\partial}{\partial\mathcal{A}^{q}{}_{r}}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}=0. \tag{B.3}$$

The vacuum functions $\Phi_{00}(G)$ in $G\in SO(2N+1)$ and $\Phi_{00}(g)$ in $g\in SO(2N)$ satisfy

$$\boldsymbol{c}_{\boldsymbol{\alpha}}\Phi_{00}(G)=\left(\boldsymbol{E}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}+\frac{1}{2}\delta_{\alpha\beta}\right)\Phi_{00}(G)=\boldsymbol{E}_{\boldsymbol{\alpha\beta}}\Phi_{00}(G)=0,\quad \Phi_{00}(1_{2N+1})=1, \tag{B.4}$$

$$\left(\boldsymbol{e}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}+\frac{1}{2}\delta_{\alpha\beta}\right)\Phi_{00}(g)=\boldsymbol{e}_{\boldsymbol{\alpha\beta}}\Phi_{00}(g)=0,\quad \Phi_{00}(1_{2N})=1. \tag{B.5}$$

By using the $SO(2N+2)$ Lie operators E^{pq}, the expression (2.22) for the $SO(2N+1)$ WF $|\,G\!>$ is converted to a form quite similar to the $SO(2N)$ WF $|\,g\!>$ as

$$|\,G\!>\,=\,<\!0\,|U(G)|\,0\!>\exp\left(\frac{1}{2}\cdot\mathcal{Q}_{pq}E^{pq}\right)|\,0\!>, \tag{B.6}$$

where we have used the nilpotency relation $(E^{\alpha 0})^2=0$. Equation (B.6) leads to the property $U(G)|\,0\!>=U(\mathcal{G})|\,0\!>$. On the other hand, from (3.15) we get

$$\det\mathcal{A}=\frac{1+z}{2}\det a,\quad \det\mathcal{B}=\left\{\frac{1-z}{2}+\frac{1}{2(1+z)}\left(x^{\mathrm{T}}q^{-1}x-x^{\dagger}x\right)\right\}\det b=0. \tag{B.7}$$

Then we obtain the vacuum function $\Phi_{00}(\mathcal{G})$ expressed in terms of the Kähler potential as

$$\overline{<\!0\,|U(\mathcal{G})|\,0\!>}=\Phi_{00}(\mathcal{G})=\left[\det(\bar{\mathcal{A}})\right]^{\frac{1}{2}}=e^{-\frac{1}{4}\mathcal{K}(\mathcal{Q},\,\mathcal{Q}^{\dagger})}e^{-i\frac{\tau}{2}}, \tag{B.8}$$

$$\Phi_{00}(\mathcal{G})=\Phi_{00}(G)=\sqrt{\frac{1+z}{2}}\,[\det(\bar{a})]^{\frac{1}{2}}=\sqrt{\frac{1+z}{2}}e^{-\frac{1}{4}\mathcal{K}(q,\,q^{\dagger})}e^{-i\frac{\tau}{2}}. \tag{B.9}$$

C Differential forms for bosons over SO(2N+2)/U(N+1) coset space

Using the differential formulas (A.14), the fermion $SO(2N+2)$ Lie operators are mapped into the regular representation consisting of functions on the coset space $\frac{SO(2N+2)}{U(N+1)}$. The boson images of the fermion $SO(2N+2)$ Lie operators $\boldsymbol{\mathcal{E}^p{}_q}$ etc. can be represented by the closed first order differential forms over the $\frac{SO(2N+2)}{U(N+1)}$ coset space in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables $\mathcal{Q}_{pq}$ and the phase variable $\tau \left(= \frac{i}{2} \ln \left[\frac{\det(A^*)}{\det(A)}\right]\right)$ of the $U(N+1)$, which is identical with the phase variable $\tau \left(= \frac{i}{2} \ln \left[\frac{\det(a^*)}{\det(a)}\right]\right)$ of the $U(N)$ due to the first equation of (B.7), as

$$\left.\begin{aligned} \boldsymbol{\mathcal{E}^p{}_q} &= \bar{\mathcal{Q}}_{pr} \frac{\partial}{\partial \bar{\mathcal{Q}}_{qr}} - \mathcal{Q}_{qr} \frac{\partial}{\partial \mathcal{Q}_{pr}} - i\delta_{pq} \frac{\partial}{\partial \tau}, \\ \boldsymbol{\mathcal{E}_{pq}} &= \mathcal{Q}_{pr} \mathcal{Q}_{sq} \frac{\partial}{\partial \mathcal{Q}_{rs}} - \frac{\partial}{\partial \bar{\mathcal{Q}}_{pq}} - i\mathcal{Q}_{pq} \frac{\partial}{\partial \tau}, \end{aligned}\right\} \tag{C.1}$$

which are derived in a way quite analogous to the $SO(2N)$ case of the fermion Lie operators. The images of the fermion $SO(2N+1)$ Lie operators are represented with the aid of those of the $SO(2N+2)$ operators. From (C.1), we can get the representations of the $SO(2N+1)$ Lie operators in terms of the variables $q_{\alpha\beta}$ and r_α [17]:

$$\left.\begin{aligned} \boldsymbol{E}^\alpha{}_\beta &= \boldsymbol{\mathcal{E}^\alpha{}_\beta} = \boldsymbol{e^\alpha{}_\beta} + \bar{r}_\alpha \frac{\partial}{\partial \bar{r}_\beta} - r_\beta \frac{\partial}{\partial r_\alpha}, \;\; \boldsymbol{e^\alpha{}_\beta} = \bar{q}_{\alpha\gamma} \frac{\partial}{\partial \bar{q}_{\beta\gamma}} - q_{\beta\gamma} \frac{\partial}{\partial q_{\alpha\gamma}} - i\delta_{\alpha\beta} \frac{\partial}{\partial \tau}\;, \\ \boldsymbol{E}_{\alpha\beta} &= \boldsymbol{\mathcal{E}_{\alpha\beta}} = \boldsymbol{e_{\alpha\beta}} + (r_\alpha q_{\beta\xi} - r_\beta q_{\alpha\xi}) \frac{\partial}{\partial r_\xi}, \;\; \boldsymbol{e_{\alpha\beta}} = q_{\alpha\gamma} q_{\delta\beta} \frac{\partial}{\partial q_{\gamma\delta}} - \frac{\partial}{\partial \bar{q}_{\alpha\beta}} - iq_{\alpha\beta} \frac{\partial}{\partial \tau}, \end{aligned}\right\} \tag{C.2}$$

$$\boldsymbol{c_\alpha} = \boldsymbol{\mathcal{E}_{0\alpha}} - \boldsymbol{\mathcal{E}^0{}_\alpha} = \frac{\partial}{\partial \bar{r}_\alpha} + \bar{r}_\xi \frac{\partial}{\partial \bar{q}_{\alpha\xi}} + (r_\alpha r_\xi - q_{\alpha\xi}) \frac{\partial}{\partial r_\xi} - q_{\alpha\xi} r_\eta \frac{\partial}{\partial q_{\xi\eta}} + ir_\alpha \frac{\partial}{\partial \tau}, \;\; \boldsymbol{c^\dagger_\alpha} = -\boldsymbol{\bar{c}_\alpha}. \tag{C.3}$$

The vacuum function $\Phi_{00}(G)$ in $G \in SO(2N+1)$ is given in (B.4). Using the relations

$$\boldsymbol{c_\alpha} \Phi_{00}(G) = 0, \quad \boldsymbol{c^\dagger_\alpha} \Phi_{00}(G) = \bar{r}_\alpha \Phi_{00}(G), \tag{C.4}$$

and the property $U(\mathcal{G})|\,0\!>\, = U(G)|\,0\!>$ ((B.6) and (B.9)), we have a relation

$$\boldsymbol{c_\alpha} U(\mathcal{G})|\,0\!> = -q_{\alpha\xi} r_\eta c^\dagger_\xi c^\dagger_\eta \cdot U(G)|\,0\!> - \left(r_\alpha + q_{\alpha\xi} c^\dagger_\xi\right) \cdot \bar{\Phi}_{00}(G) \exp\left(\frac{1}{2} \cdot q_{\gamma\delta} E^{\gamma\delta}\right) |\,0\!>, \tag{C.5}$$

from which we easily get the exact identities

$$\boldsymbol{c_\alpha} U(\mathcal{G})|\,0\!> = \left(-r_\alpha + r_\alpha r_\xi c^\dagger_\xi - q_{\alpha\xi} c^\dagger_\xi\right) \cdot U(G)|\,0\!>, \quad \boldsymbol{c^\dagger_\alpha} U(\mathcal{G})|\,0\!> = -c^\dagger_\alpha \cdot U(G)|\,0\!>. \tag{C.6}$$

Successively using these identities, on the $U(\mathcal{G})|\,0\!>$, the operators $\boldsymbol{c_\alpha}$ and $\boldsymbol{c^\dagger_\alpha}$ are shown to satisfy exactly the anti-commutation relations of the fermion annihilation-creation operators [22]:

$$\left(\boldsymbol{c^\dagger_\alpha c_\beta} + \boldsymbol{c_\beta c^\dagger_\alpha}\right) U(\mathcal{G})|\,0\!> = \delta_{\alpha\beta} \cdot U(\mathcal{G})|\,0\!>, \tag{C.7}$$

$$(\boldsymbol{c_\alpha c_\beta} + \boldsymbol{c_\beta c_\alpha})\, U(\mathcal{G})|\,0\!> = (\boldsymbol{c^\dagger_\alpha c^\dagger_\beta} + \boldsymbol{c^\dagger_\beta c^\dagger_\alpha}) U(\mathcal{G})|\,0\!> = 0. \tag{C.8}$$

D Euler-Lagrange equation of motion for Hamiltonian

The expectation values of the fermion Lie operators by the $|\,G\!>$ are calculated easily as

$$<\!E^{\alpha}{}_{\beta} + \frac{1}{2}\delta_{\alpha\beta}\!>_{G(t)} = \bar{\mathcal{R}}_{\alpha\beta}, \quad <\!E_{\alpha\beta}\!>_{G(t)} = -\mathcal{K}_{\alpha\beta}, \quad <\!E^{\alpha\beta}\!>_{G(t)} = \bar{\mathcal{K}}_{\alpha\beta}, \tag{D.1}$$

$$<\!c_{\alpha}\!>_{G(t)} = \mathcal{K}_{\alpha 0} - \bar{\mathcal{R}}_{\alpha 0}, \quad <\!c^{\dagger}_{\alpha}\!>_{G(t)} = \bar{\mathcal{K}}_{\alpha 0} - \mathcal{R}_{\alpha 0}, \tag{D.2}$$

where matrix elements $\mathcal{R}_{pq}$ and $\mathcal{K}_{pq}$ are given in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variable $\mathcal{Q}_{pq}$ as

$$\mathcal{R}_{pq} = \left[\mathcal{Q}(1_{N+1} + \mathcal{Q}^{\dagger}\mathcal{Q})^{-1}\mathcal{Q}^{\dagger}\right]_{pq} (\mathcal{R}^{\dagger} = \mathcal{R}), \quad \mathcal{K}_{pq} = \left[\mathcal{Q}(1_{N+1} + \mathcal{Q}^{\dagger}\mathcal{Q})^{-1}\right]_{pq} (\mathcal{K}^{\mathrm{T}} = -\mathcal{K}). \tag{D.3}$$

Using the above expressions, the following relations and derivative formulas are easily proved:

$$\mathcal{Q}(1_{N+1} - \mathcal{R}) = (1_{N+1} - \bar{\mathcal{R}})\mathcal{Q} = \mathcal{K}, \quad \bar{Q}\mathcal{K} = \bar{\mathcal{K}}\mathcal{Q} = \mathcal{R}, \tag{D.4}$$

$$\left.\begin{aligned}
\frac{\partial \mathcal{R}_{pq}}{\partial \mathcal{Q}_{uv}} &= -\bar{\mathcal{K}}_{pu}(1_{N+1} - \mathcal{R})_{vq} + \bar{\mathcal{K}}_{pv}(1_{N+1} - \mathcal{R})_{uq}, \\
\frac{\partial \mathcal{R}_{pq}}{\partial \bar{\mathcal{Q}}_{uv}} &= -(1_{N+1} - \mathcal{R})_{pu}\mathcal{K}_{vq} + (1_{N+1} - \mathcal{R})_{pv}\mathcal{K}_{uq}, \\
\frac{\partial \mathcal{K}_{pq}}{\partial \mathcal{Q}_{uv}} &= (1_{N+1} - \bar{\mathcal{R}})_{pu}(1_{N+1} - \mathcal{R})_{vq} - (1_{N+1} - \bar{\mathcal{R}})_{pv}(1_{N+1} - \mathcal{R})_{uq}, \\
\frac{\partial \mathcal{K}_{pq}}{\partial \bar{\mathcal{Q}}_{uv}} &= \mathcal{K}_{pu}\mathcal{K}_{vq} - \mathcal{K}_{pv}\mathcal{K}_{uq}.
\end{aligned}\right\} \tag{D.5}$$

Using the property $U(G)|\,0\!> = U(\mathcal{G})|\,0\!>$ $(G \in SO(2N+1),\ \mathcal{G} \in SO(2N+2))$ and (B.6), a classical Lagrangian of a system embedded into the $SO(2N+2)$ group is given as

$$\begin{aligned}
L(\mathcal{G}(t)) &= <\!0\,|U^{\dagger}(\mathcal{G}(t))\left\{i\hbar\frac{\partial}{\partial t} - H\right\}U(\mathcal{G}(t))|\,0\!> \\
&= \frac{i\hbar}{2}\mathrm{tr}\{(1 + \mathcal{Q}^{\dagger}\mathcal{Q})^{-1}(\mathcal{Q}^{\dagger}\dot{\mathcal{Q}} - \dot{\mathcal{Q}}^{\dagger}\mathcal{Q})\} - \frac{\hbar}{2}\dot{\tau} - <\!H\!>_{G(t)}\ .
\end{aligned} \tag{D.6}$$

The Euler-Lagrange EOM for the $\frac{SO(2N+2)}{U(N+1)}$ coset variables is calculated to be

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{\mathcal{Q}}}}\right) - \frac{\partial L}{\partial \bar{\mathcal{Q}}} = -\frac{1}{2}i\hbar\left[\dot{\mathcal{Q}}(1 + \mathcal{Q}^{\dagger}\mathcal{Q})^{-1} + (1 + \mathcal{Q}\mathcal{Q}^{\dagger})^{-1}\dot{\mathcal{Q}}\right. \\
&\left.-\mathcal{Q}(1 + \mathcal{Q}^{\dagger}\mathcal{Q})^{-1}\mathcal{Q}^{\dagger}\dot{\mathcal{Q}}(1 + \mathcal{Q}\mathcal{Q}^{\dagger})^{-1} - (1 + \mathcal{Q}\mathcal{Q}^{\dagger})^{-1}\dot{\mathcal{Q}}\mathcal{Q}^{\dagger}(1 + \mathcal{Q}\mathcal{Q}^{\dagger})^{-1}\mathcal{Q}\right] + \frac{\partial\!<\!H\!>_G}{\partial \bar{\mathcal{Q}}} = 0,
\end{aligned} \tag{D.7}$$

and its complex conjugate. The Hamiltonian H consists of one-body and two-body operators. From (D.7), we obtain the Euler-Lagrange EOM for the variable $\mathcal{Q}$ as

$$\left.\begin{aligned}
&\dot{\mathcal{Q}} = -\frac{i}{\hbar}\left(1 - \bar{\mathcal{R}}\right)^{-1}\frac{\left(\partial\!<\!H\!>_G + \frac{1}{2}M_{\alpha}\!<\!c^{\dagger}_{\alpha}\!>_G + \frac{1}{2}\bar{M}_{\alpha}\!<\!c_{\alpha}\!>_G\right)}{\partial \bar{\mathcal{Q}}}\left(1 - \mathcal{R}\right)^{-1}, \\
&\langle H\rangle_G = h_{\alpha\beta}\mathcal{R}_{\alpha\beta} + \frac{1}{2}[\alpha\beta|\gamma\delta]\left(\mathcal{R}_{\alpha\beta}\mathcal{R}_{\gamma\delta} - \frac{1}{2}\bar{\mathcal{K}}_{\alpha\gamma}\mathcal{K}_{\delta\beta}\right),\ M_{\alpha} = k_{\alpha\beta}\!<\!c_{\beta}\!>_G + l_{\alpha\beta}\!<\!c^{\dagger}_{\beta}\!>_G,
\end{aligned}\right\} \tag{D.8}$$

in which a classical Hamiltonian function accompanies additional terms appearing as a classical part of Lagrange multiplier terms $k_{\alpha\beta}$ and $l_{\alpha\beta}$ to select out a spinor sub-space. Using (D.5), (D.8) is transformed into an EOM for HB amplitudes which involve effects of unpaired modes as well as of paired modes. With the aid of the generalized density matrix (5.11), we thus derive the ETDHB equation in which both modes are treated in a unified way [14, 17, 22].

E Derivation of (5.48) and (5.49)

Using the representation for $\mathcal{Q}_f$ given in (5.44), the inverse of the matrix $\mathcal{X}_f$, $\mathcal{X}_f^{-1}$, is expressed as

$$\mathcal{X}_f^{-1} = \begin{bmatrix} \chi_f^{-1} + r_f r_f^\dagger & -f q \bar{r}_f \\ -f r_f^{\mathrm{T}} q^\dagger & 1 + r_f^\dagger r_f \end{bmatrix}, \quad \chi_f^{-1} = 1_N + f^2 q q^\dagger, \tag{E.1}$$

from which the inverse matrix $\mathcal{X}_f$ $(= [1_{N+1} + f^2 \mathcal{Q}_f \mathcal{Q}_f^\dagger]^{-1})$ in (5.47) is given as

$$\mathcal{X}_f = \begin{bmatrix} \mathcal{Q}_{fqq^\dagger} & \mathcal{Q}_{fqr} \\ \mathcal{Q}_{fqr}^\dagger & \mathcal{Q}_{fr^\dagger r} \end{bmatrix}, \quad \begin{aligned} \mathcal{Q}_{fqq^\dagger} &= \left[\chi_f^{-1} + r_f r_f^\dagger - \frac{f^2}{1 + r_f^\dagger r_f} q \bar{r}_f r_f^{\mathrm{T}} q^\dagger \right]^{-1}, \\ \mathcal{Q}_{fqr} &= \frac{f}{1 + r_f^\dagger r_f} \mathcal{Q}_{fqq^\dagger}^\dagger q \bar{r}_f, \\ \mathcal{Q}_{fr^\dagger r} &= \frac{f}{1 + r_f^\dagger r_f} \left(1 + f r_f^{\mathrm{T}} q^\dagger \mathcal{Q}_{fqr} \right). \end{aligned} \tag{E.2}$$

Let us introduce an $N \times N$ matrix Y_f defined in the form

$$Y_f = \chi_f r_f r_f^\dagger - \frac{f^2}{1 + r_f^\dagger r_f} \chi_f q \bar{r}_f r_f^{\mathrm{T}} q^\dagger. \tag{E.3}$$

The $\mathcal{Q}_{fqq^\dagger}$ in (E.2) is related to Y_f by

$$\mathcal{Q}_{fqq^\dagger} = (1_N + Y_f)^{-1} \chi_f. \tag{E.4}$$

By repeated uses of both the relation and the identity

$$r_f^\dagger \chi_f r_f = \frac{1 - Z^2}{Z^2}, \quad r_f^{\mathrm{T}} q^\dagger \chi_f q \bar{r}_f = \frac{1}{f^2} \left(1 + r_f^\dagger r_f - \frac{1}{Z^2} \right), \quad r_f^{\mathrm{T}} q^\dagger \chi_f r_f = r_f^\dagger \chi_f q \bar{r}_f = 0, \tag{E.5}$$

we can compute Y_f^n $(n \geq 1)$ as

$$Y_f^n = \left(\frac{1 - Z^2}{Z^2} \right)^{n-1} \chi_f r_f r_f^\dagger - (-1)^{n-1} \left(1 - \frac{1}{1 + r_f^\dagger r_f} \frac{1}{Z^2} \right)^{n-1} \frac{f^2}{1 + r_f^\dagger r_f} \chi_f q \bar{r}_f r_f^{\mathrm{T}} q^\dagger, \tag{E.6}$$

Using the formula for an infinite summation, an inverse matrix $(1_N + Y_f)^{-1}$ is calculated as

$$\begin{aligned} (1_N + Y_f)^{-1} &= 1_N + \textstyle\sum_{n=1}^\infty (-1)^n Y_f^n \\ &= 1_N - \textstyle\sum_{n=0}^\infty \left(-\frac{1 - Z^2}{Z^2} \right)^n \chi_f r_f r_f^\dagger + \textstyle\sum_{n=0}^\infty \left(1 - \frac{1}{1 + r_f^\dagger r_f} \frac{1}{Z^2} \right)^n \frac{f^2}{1 + r_f^\dagger r_f} \chi_f q \bar{r}_f r_f^{\mathrm{T}} q^\dagger \\ &= 1_N - Z^2 \chi_f r_f r_f^\dagger + Z^2 f^2 \chi_f q \bar{r}_f r_f^{\mathrm{T}} q^\dagger. \end{aligned} \tag{E.7}$$

The geometric series expansion appearing in the right-hand side of the first and second lines of (E.7) converge only if Z^2 is sufficiently close to 1. However, by direct calculation of $(1_N + Y_f)(1_N + Y_f)^{-1}$ using (E.3), the last equation of (E.7) and (E.5), it is proved to be 1_N. Then one can see that the expression obtained for the matrix $(1_N + Y_f)^{-1}$ in (E.7) is indeed true for all $0 \leq Z^2 \leq 1$. Substituting (E.7) into (E.4) and (E.2), we can get (5.48) and (5.49). If we put $f = 1$, we also get (5.17) and (5.18).

References

[1] B. Zumino, Phys. Lett. **87B** (1979), 203.

[2] S. Groot Nibbelink, T.S. Nyawelo and J.W. van Holten, Nucl. Phys. **B594** (2001), 441.

[3] K. Higashijima, T. Kimura and M. Nitta, Nucl. Phys. **B623** (2002), 133.

[4] U. Lindström and M. Roček, Nucl. Phys. **222** (1983), 285.
N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Comm. Math. Phys. **108** (1987), 535.

[5] M. Arai, M. Naganuma, M. Nitta and N. Sakai, Nucl. Phys. **B652** (2003), 35.

[6] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108** (1957), 1175.

[7] N. N. Bogoliubov, Soviet Phys. Uspekhi, **67** (1959), 236.

[8] P. Ring and P. Schuck, *The nuclear many-body problem*, Springer, Berlin, 1980.

[9] J. P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems*, The MIT Press, Cambridge, Massachusetts, 1986.

[10] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, 1986; Phys. Rep. **146** (1987), 135, and references therein.

[11] S. Nishiyama, Prog. Theor. Phys. **66** (1981), 348; **68** (1982), 680.

[12] F. Delduc and G. Valent, Nucl. Phys. **B253** (1985), 494.

[13] J.W. van Holten, Z. Phys. **C27** (1985), 57; Nucl. Phys. **B260** (1985), 125.
Y. Achiman, S. Aoyama and J.W. van Holten, Phys. Lett. **150B** (1985), 153.
S. Aoyama, Z. Phys. **C32** (1986), 113.

[14] H. Fukutome, M. Yamamura and S. Nishiyama, Prog. Theor. Phys. **57** (1977), 1554.

[15] J. Schwinger, in *Quantum Theory of Angular Momentum*, eds. L. C. Biedenharn and H. van Dam (Academic Press, New Yor, 1965), p. 229.

[16] M. Yamamura and S. Nishiyama, Prog. Theor. Phys. **56** (1976), 124.

[17] H. Fukutome, Prog. Theor. Phys. **58** (1977), 1692.
H. Fukutome and S. Nishiyama, Prog. Theor. Phys. **72** (1984), 239.

[18] H. Fukutome, Int. J. Quantum Chem. **29** (1981), 955.
H. Fukutome, Prog. Theor. Phys. **65** (1981), 809.

[19] F.D. Murnahghan, *The Theory of Group Representations*, John Hopkins Press, Baltimore, 1938.

[20] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia Univ. Press, New York, 1954.

[21] K. Higashijima and M. Nitta, Prog. Theor. Phys. **105** (2001), 243.

[22] S. Nishiyama, Int. J. Mod. Phys. **E7** (1998), 677.